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Research Article:

Solving Large Scale System of Fuzzy Linear Equations Based on Multi-objective Optimization Technique

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Abstract

This paper considers solving a large-scale system of fuzzy linear equations that arise in various applications such as supply chain, transport, and networks (water, electricity, and gas), etc. The significance of applications containing ambiguous data necessitates the development of methodologies to address it. Such type of problem cannot be solved accurately by the analytical methods or the classical numerical iterative method. Here we will adapt a technique that transforms the fuzzy linear system into a multi-objective optimization problem. Then the obtained multi-objective optimization problem will be transformed into a single objective optimization problem. The Leap Frog method is one of the head methods that are used to solve single optimization problems. It will be introduced as a part of the introduced technique for solving the obtained single optimization problem. The convexity and the convergence for the approach and the Leap Frog method will be present. Some test examples are introduced to ensure the accuracy of the proposed approach.

1. Introduction

Most real-life problems involve uncertain data in their performance, such as planning, scheduling, and decision-making. The fuzzy number is one type that represents the uncertainty data in the problems.

Zadeh, the first one, introduced the fuzzy theory in (Zadeh, 1965). There are many articles used and dealt with the concepts of fuzzy numbers as (Arabi Nowdeh et al., 2019; Bellman & Zadeh 1970; Bozanic et al., 2022; Garg & Rani, 2022; Khalili Nasr et al., 2021; Lourenço & César, 2022; Rogers, 2019; Warid et al., 2016). The issues of bipolar fuzzy linear systems and bipolar fuzzy complex linear systems are studied in (Akram et al., 2022). The existence and uniqueness of the solution for fully fuzzy lin-

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ear systems with trapezoidal and hexagonal fuzzy numbers were investigated in (Ziqan et al., 2022). Finding the pareto set of solutions for a fully fuzzy multi-objective transportation problem with the conditions of uncertainty considered in (Niksirat, 2022). A fuzzy decision variable framework for large-scale multiobjective optimization was investigated in (Xu et al., 2023). An algorithm for solving the fully fuzzy multi-objective linear fractional (FFMOLF) optimization problem (Arya et al., 2020). Many applications and problems can be formulated as a fuzzy linear system, and it will be very complicated when the system is large scale. One of these applications is predicting and monitoring the quality of air, which is very needed to control and identify the adverse health effects because the low-quality (Rao et al., 2024) can be formed in a fuzzy system model to obtain very accurate results. An enhanced neural network used for classification, including intuitive, interpretable correlated-contours fuzzy rules (Patro et al., 2023).

Moreover, solving the large-scale fuzzy system's problems is a hard job and is impossible to solve analytically. The accuracy of the classical numerical methods for solving large-scale systems decreases more rapidly when the size of the system increases. Hence, one can go to alternative techniques to solve the large-scale fuzzy system. Reformulation of the large-scale fuzzy system into a multi-objective, unconstrained optimization problem is a good choice to solve it.

Multi-objective optimization is concerned with optimizing problems of two or more conflicting objectives subject simultaneously to certain constraints. There are various fields that contain multi-objective optimization problems: aircraft design, finance, automobile design, the oil and gas industry, wind farm design, or wherever optimal solutions need to be found in the manner of trade-offs between conflict objectives. Methods of solving multi-objective optimization problems updated in some ways such as: artificial intelligence methods, transforming the multi-objective optimization problem to a single objective optimization problem, etc.

In this study, we are concerned with solving those important problems that contain fuzzy variables. So, we will depend on the way of transforming the multi-objective optimization problem into an unconstrained single objective optimization problem. Then we use one of the successful methods for solving unconstrained optimization problems, the Leap-Frog method created by (Snyman, 1989). The Leap-Frog method is one of the first-degree methods that depends on the gradient and the function evaluation.

Some points that this study contributes can be summarized as:

a. Solving the large number of applications that can be modeled in fuzzy form.
b. Dealing with fuzzy variable problems remains at the top trend.
c. Solving problems that have a huge number of fuzzy variables.
d. Finding an accurate solution for the systems with fuzzy data.

This paper is organized as follows: Section 2 previews some of the elementary and definitions of the fuzzy numbers. In Section 3, we introduce the fuzzy linear system. The proposed approach will be in Section 4. Convexity and convergence are discussed in Section 5. Numerical results for some examples are presented in Section 6. Finally, the conclusion is in Section 7.

2. Preliminaries

In this section, we present some elementary that will be used next.

The parametric form of an arbitrary fuzzy number is represented as an ordered pair of functions $(\underline{s}(r), \overline{s}(r))$, $0 \le r \le 1$ and satisfy the following conditions: (Goetschel & Voxman, 1986; Ma et al., 1999):

1) $\underline{s}(r)$ is a left-continuous bounded non-decreasing function over $[0,1]$,

2) $\bar{s}(r)$ is a left-continuous bounded non-increasing function over $[0,1]$,
3) $\underline{s}(r) \leq \bar{s}(r)$, $0 \leq r \leq 1$.

The fuzzy numbers space, $\{\underline{s}(r), \bar{s}(r)\}$, by appropriate definitions, becomes a convex cone E^1. The addition and the scalar multiplication of fuzzy numbers for any two arbitrary fuzzy numbers $s = (\underline{s}, \bar{s})$, $t = (\underline{t}, \bar{t}) \in E^1, k \in R$ are Dubois & Prade (1980):

$$(\underline{s+t})(r) = \underline{s}(r) + \underline{t}(r), \quad (\overline{s+t})(r) = \bar{s}(r) + \bar{t}(r). \tag{1}$$

$$\begin{cases} (\underline{ks})(r) = k\underline{s}(r), (\overline{ks})(r) = k\bar{s}(r) \ \text{ if } k \geq 0, \\ (\underline{ks})(r) = k\bar{s}(r), (\overline{ks})(r) = k\underline{s}(r) \ \text{ if } k < 0. \end{cases} \tag{2}$$

Definition 2.1. (Bazaraa et al. ,2006) If the function $f(U): S \to R$, where S is a nonempty convex set in R^n. Then $f(U)$ is called convex on S if:

$$f(\mu U_1 + (1-\mu)U_2) \leq \mu f(U_1) + (1-\mu) f(U_2), \tag{3}$$

for each $U_1, U_2 \in S$ and for each $\mu \in (0,1)$. The function f is said to be strictly convex on S if the inequality is true as a strict inequality.

3. Fuzzy linear system of equations

The fuzzy linear system of equations (FLSE) is defined as in the following definition.

Definition 3.1. (Friedman et al., 1998) The system of the form,

$$\begin{aligned} a_{11}u_1 + a_{12}u_2 + \ldots + a_{1n}u_n &= v_1, \\ a_{21}u_1 + a_{22}u_2 + \ldots + a_{2n}u_n &= v_2, \\ &\vdots \\ a_{n1}u_1 + a_{n2}u_2 + \ldots + a_{nn}u_n &= v_n, \end{aligned} \tag{4}$$

such that the coefficients matrix $A = (a_{ij})$, $i,j = 1,\ldots,n$ is $n \times n$ crisp matrix and $v_i \in E^1, i = 1,\ldots,n$ is called a fuzzy system of linear equations.

Definition 3.2. (Abbasbandy & Jafarian, 2006) A fuzzy number vector $(u_1, u_2, \ldots, u_n)^T$ given as $u_i = (\underline{u}_i(r), \bar{u}_i(r))$, $i = 1,\ldots,n$, $0 \leq r \leq 1$ is called a solution of Eq. (4) if,

$$\sum_{j=1}^{n} \underline{a_{ij}u_j} = \underline{v}_i, \tag{5}$$

$$\sum_{j=1}^{n} \overline{a_{ij}u_j} = \overline{v}_i. \tag{6}$$

The systems of equations (5) and (6) can be converted to the system:

$$DU = V, \tag{7}$$

where, $U = (\underline{u}_1(r), \underline{u}_2(r), \ldots, \underline{u}_n(r), -\bar{u}_1(r), -\bar{u}_2(r), \ldots, -\bar{u}_n(r))^T$,$V = (\underline{v}_1(r), \underline{v}_2(r), \ldots, \underline{v}_n(r), -\bar{v}_1(r), -\bar{v}_n(r), \ldots, -\bar{v}_n(r))^T$ and $D = (d_{lm})$, $l,m = 1,\ldots,p$ is $p \times p$ non-negative crisp matrix, $p = 2n$ and d_{lm} compute as:

$$d_{lm} = \begin{cases} aij, \ \text{if } aij > 0 \ \text{and } \ l,m = 1,2,\ldots,n, \\ -aij, \ \text{if } aij < 0 \ \text{and } l = 1,2,\ldots,n, m = n+1, n+2,\ldots,p, \\ -aij, \ \text{if } \ aij < 0 \ \text{and } \ l = n+1, n+2,\ldots,p, \ m = 1,2,\ldots,n, \\ aij, \ \text{if } \ aij > 0 \ \text{and } \ l,m = n+1, n+2,\ldots,p, \\ 0, \ \text{otherwise.} \end{cases}$$

Definition 3.3. (Friedman et al.,1998) If $U = \{(\underline{u}_i(r), \bar{u}_i(r)), i = 1,\ldots,n\}$ is the unique solution of $DU = V$ and $(\underline{v}_i(r), \bar{v}_i(r))$ are linear functions of r, then the fuzzy number $T = \{(\underline{t}_i(r), \bar{t}_i(r)), i = 1,\ldots,n\}$ defined as: $\underline{t}_i(r) = \min\{\underline{u}_i(r), \bar{u}_i(r), \underline{u}_i(1), \bar{u}_i(1)\}$, $\bar{t}_i(r) = \max\{\underline{u}_i(r), \bar{u}_i(r), \underline{u}_i(1), \bar{u}_i(1)\}$ is the fuzzy solution of $DU = V$.

Definition 3.4. In addition to definition 3 if $(\underline{u}_i(r), \bar{u}_i(r)), i = 1,\ldots,n$ are all fuzzy numbers, then $\underline{t}_i(r) = \underline{u}_i(r)$, $\bar{t}_i(r) = \bar{u}_i(r)$ and T is a strong fuzzy solution. Otherwise T is a weak fuzzy solution.

Theorem 3.5. Friedman et al. (1998) The unique solution U of (7) is always a fuzzy vector for arbitrary vector V, if and only if D^{-1} is non-negative for the non-singular matrix D.

4. The Proposed strategy

Our approach depends on transforming the system of equations (7) to be an unconstrained multi-objective optimization problem such as:

$$F = DU - V, \quad \text{where} \quad F = (f_1, f_2, \ldots, f_p)^T. \quad (8)$$

In general, there are many strategies for solving the multi-objective optimization problem (MOOP). One of the appropriate strategies is the weight-sum method. Here we aim to adapt this technique to transform the MOOP to a single objective optimization problem (SOOP). Since the system under study is a linear system, the summing of the objective functions with weight will determine the extreme points; there is no guarantee of finding the minimum point. Miettinen (1998). In another view, it is clear that the system of equations (8) transformed to be seeking to minimize the error. So, we will be based on the square of the objectives as follows:

$$\Phi(U) = \sum_{l=1}^{p} \omega_l f_l^2(U), \quad (9)$$

where $\omega > 0$ and $\sum_{l=1}^{p} \omega_l = 1$. We will depend on equal weight for the objective functions in this study: that is, the objectives have the same importance.

The Leap Frog Algorithm (LFA) of Snyman (1989) is one of the best algorithms for solving a large-scale, unconstrained single optimization problem. Hence, it will be adapted in the proposed algorithm in order to solve equation (9). In the next we explain the main steps of the LFA. The mechanism of the LFA based on the iterations:

$$U_{k+1} = U_k + \lambda S_k, \quad k = 0,1,2,\ldots \quad (10)$$

where U_0, S_0 are guessed value and λ is constant chosen to be > 0.

$$S_k = S_{k-1} + \lambda G_k. \quad (11)$$

$$G_k = -\nabla\Phi(U_k). \quad (12)$$

Since $\lambda > 0$ the sufficient condition for descent (i.e. $\Delta\Phi_k < 0$) is

$$S_k^T G_{k+1} > 0. \quad (13)$$

So, the basis for using the LFA in the minimization problem should be tested for every step to satisfy the sufficient condition (13) for the descent. The mechanism process flow diagram is previewed in figure 1.

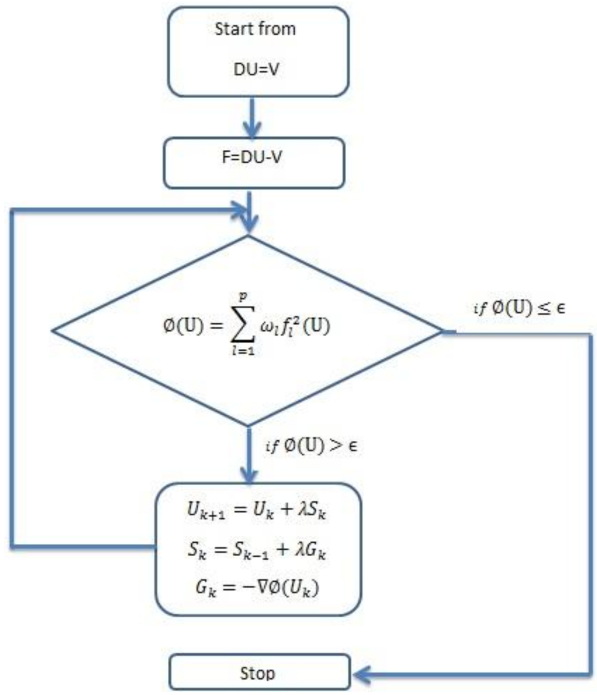

Figure 1: The process flow diagram

5. Problem convexity and convergence

Here we study the convexity of the problem (8) and the convergence of the proposed approach. based on definition (2.1), one can prove the convexity of the system, which is essentially for applying LFA, as follows:

Let we take a function $f_l \in F$ where,

$$f_l(U) = d_{l1}\underline{u}_1 + d_{l2}\underline{u}_2 + \ldots + d_{ln}\underline{u}_n - d_{l(n+1)}\overline{u_1} - \ldots - d_{lp}\overline{u_n} - v_l,$$

and let $Q = (q_1, q_2, \ldots, q_p)$, $T = (t_1, t_2, \ldots, t_p)$ are two solutions for f_l then,

$$\begin{aligned} f_l(\mu Q + (1-\mu)T) &= d_{l1}(\mu \underline{q_1} + (1-\mu)\underline{t_1}) + d_{l2}(\mu \underline{q_2} + (1-\mu)\underline{t_2}) + \ldots + d_{ln}(\mu \underline{q_n} \\ &+ (1-\mu)\underline{t_n}) - d_{l(n+1)}(\mu \overline{q_1} + (1-\mu)\overline{t_1}) - \ldots - d_{lp}(\mu \overline{q_n} \\ &+ (1-\mu)\overline{t_n}) - v_l(\mu + (1-\mu)). \\ &= \mu[d_{l1}\underline{q_1} + d_{l2}\underline{q_2} + \ldots + d_{ln}\underline{q_n} - d_{l(n+1)}\overline{q_1} - \ldots - d_{lp}\overline{q_n} - v_l] \\ &+ (1-\mu)[d_{l1}\underline{t_1} + d_{l2}\underline{t_2} + \ldots + d_{ln}\underline{t_n} - d_{l(n+1)}\overline{t_1} - \ldots - d_{lp}\overline{t_n} - v_l] \\ &= \mu f_l(Q) + (1-\mu) f_l(T). \end{aligned}$$

Hence the functions in problem (8) are all convex functions. To this end the convergence of applying LFA and the existence of at most one optimal solution are verified by theorem (4.1) in (Snyman, 1989).

6. Numerical examples

In this section we consider three examples to ensure the accurate of our approach.

Example 6.1. Studying the following FLSE

$$\begin{aligned} u_1 + 2u_2 - 3u_3 &= (-23 + 18r, 5 - 10r), \\ 2u_1 - 5u_2 + 7u_3 &= (24r, 66 - 42r), \\ -9u_1 + u_2 + 2u_3 &= (-29 + 12r, 5 - 22r). \end{aligned} \quad (14)$$

From equations (1), (2), (5), (6) the system converts to the form of equations (8) to be:

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 3 \\ 2 & 0 & 7 & 0 & 5 & 0 \\ 0 & 1 & 2 & 9 & 0 & 0 \\ 0 & 0 & 3 & 1 & 2 & 0 \\ 0 & 5 & 0 & 2 & 0 & 7 \\ 9 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} \underline{u_1} \\ \underline{u_2} \\ \underline{u_3} \\ -\overline{u_1} \\ -\overline{u_2} \\ -\overline{u_3} \end{bmatrix} - \begin{bmatrix} -23 + 18r \\ 24r \\ -29 + 12r \\ -5 + 10r \\ -66 + 42r \\ -5 + 22r \end{bmatrix}. \quad (15)$$

The results of applying the proposed approach for the equations in (15) after transformed by equation (9) are present in figure 2 and compared with the exact solutions (16):

$$\begin{aligned} u_1 &= (\underline{u_1}, \overline{u_1}) = (2 + r, 4 - r), \\ u_2 &= (\underline{u_2}, \overline{u_2}) = (1 + r, 5 - 3r), \\ u_3 &= (\underline{u_3}, \overline{u_3}) = (3 + r, 9 - 5r). \end{aligned} \quad (16)$$

The objective error (9) and the numerical error for the proposed approach present in table 1. The error for the variables u_i with $0 \le r \le 1$ is previewed in figure 3.

n	p	Objective error	$\|\|.\|\|_2$
3	6	1.8554E-27	1.1181E-13

Table 1: Execution data for example 6.1 with r = 1

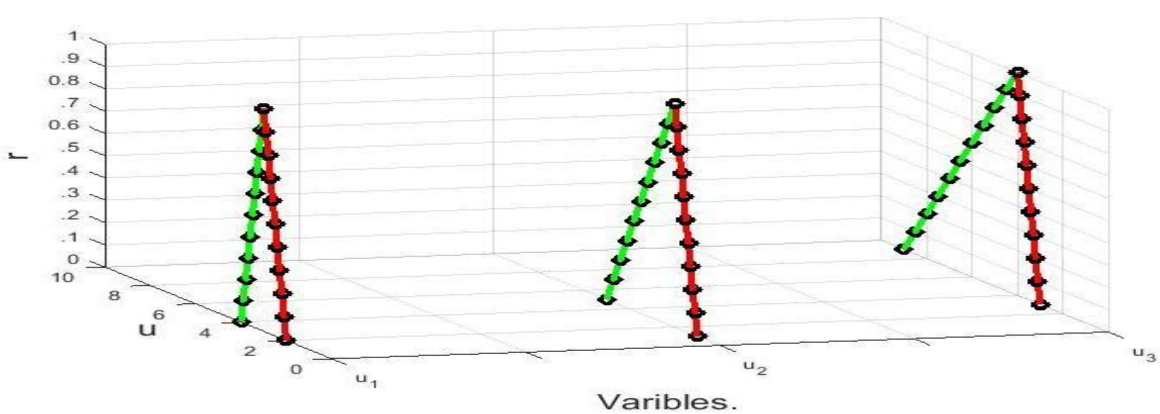

Figure 2: Exact and approximate solutions (circles for $\underline{u_i}$ and diamond for $\overline{u_i}$) for example 6.1.

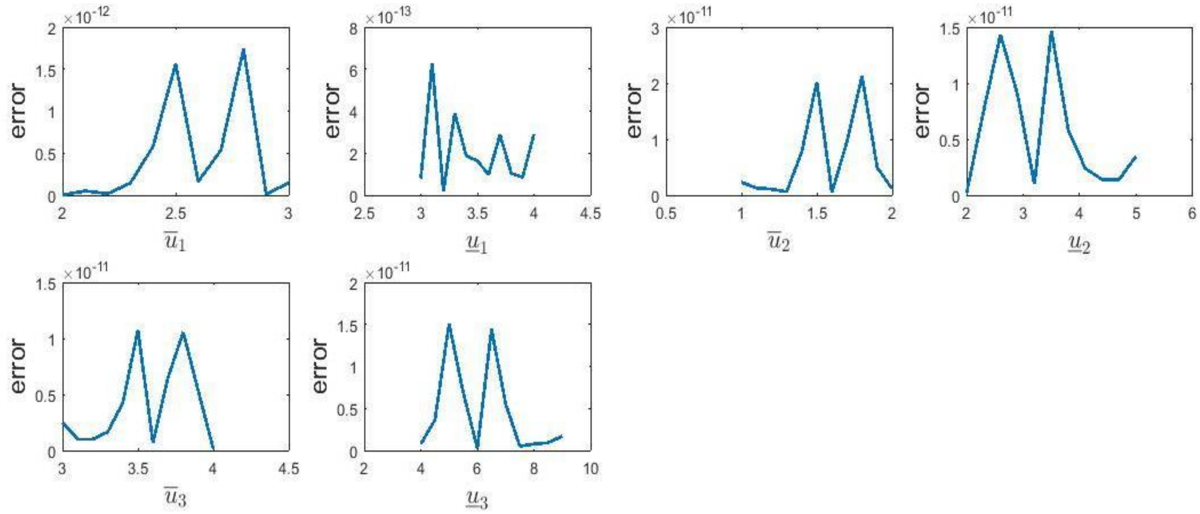

Figure 3: The error for the variables u_i for Example 6.1.

Example 6.2. In considering the following FLSE

$$\begin{aligned} &u_1 - 2u_2 + 3u_4 = (14r, 23 - 9r), \\ &-3u_1 - 5u_2 + 4u_3 = (-34 + 17r, 12 - 29r), \\ &4u_1 + 3u_2 - u_3 - 2u_4 = (-22 + 26r, 19 - 15r), \\ &-6u_3 + u_4 = (-20 + 20r, 8 - 8r). \end{aligned} \tag{17}$$

Again, we have:

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \end{bmatrix} = \left|\begin{bmatrix} 1 & 0 & 0 & 3 & 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 & 3 & 5 & 0 & 0 \\ 4 & 3 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 6 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 & 0 & 3 \\ 3 & 5 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & 2 & 4 & 3 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \\ \underline{u}_3 \\ \underline{u}_4 \\ -\overline{u}_1 \\ -\overline{u}_2 \\ -\overline{u}_3 \\ -\overline{u}_4 \end{bmatrix} - \begin{bmatrix} 14r \\ -34 + 17r \\ -22 + 26r \\ -20 + 20r \\ -23 + 9r \\ -12 + 29r \\ -19 + 15r \\ -8 + 8r \end{bmatrix}\right|. \tag{18}$$

Figure 4 preview the obtained results of the proposed approach for the equations in (18) after using equation (9), and compared with the exact solutions (19):

$$\begin{aligned} u_1 &= (\underline{u}_1, \overline{u}_1) = (-2 + 4r, 3 - r), \\ u_2 &= (\underline{u}_2, \overline{u}_2) = (2 + r, 5 - 2r), \\ u_3 &= (\underline{u}_3, \overline{u}_3) = (r, 4 - 3r), \\ u_4 &= (\underline{u}_4, \overline{u}_4) = (4 + 2r, 8 - 2r). \end{aligned} \tag{19}$$

The objective error (9) and the numerical error for the proposed approach preview in table 2. The error for the variables u_i with $0 \le r \le 1$ is preview in figure 5.

n	p	Objective error	$\|\|.\|\|_2$
4	8	5.6112E-27	2.0451E-13

Table 2: Execution data for example 6.2 with $r = 1$

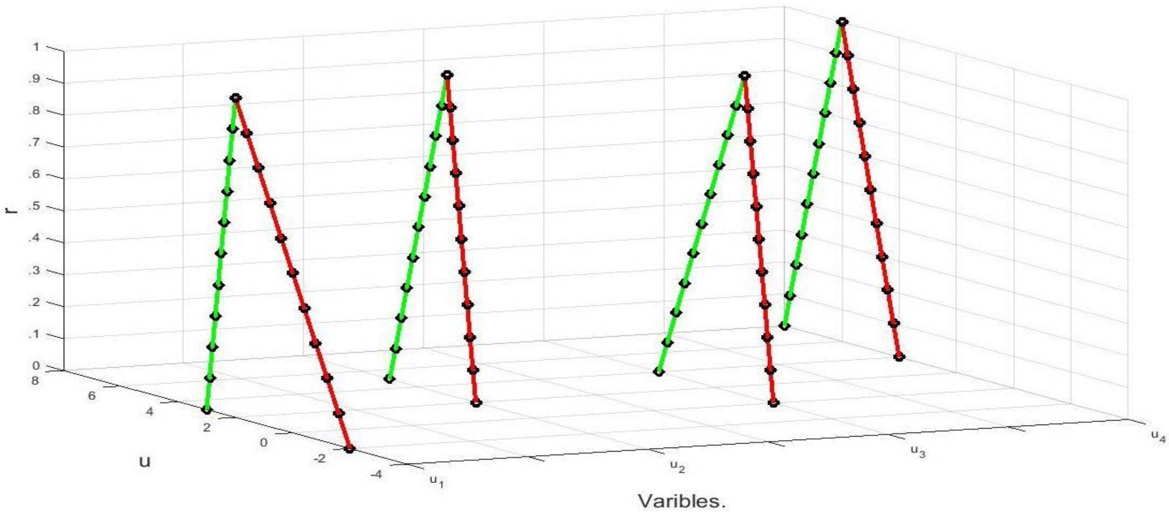

Figure 4: Exact and approximate solutions (circles for $\underline{u}_i$ and diamond for $\overline{u_i}$) for example 6.2.

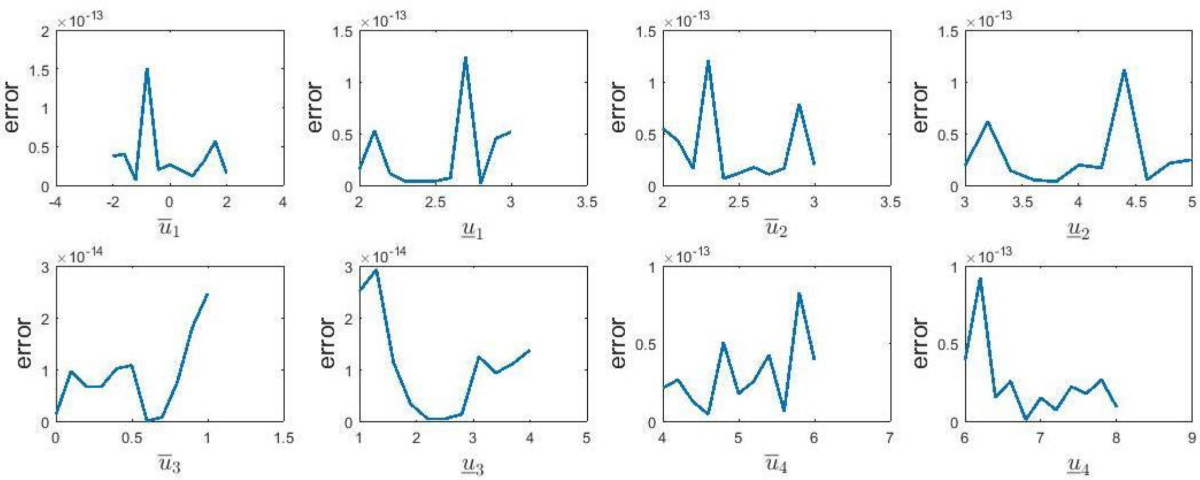

Figure 5: The error for the variables u_i for example 6.2.

Example 6.3. In considering example 5.1. in Dehghan & Hashemi (2006)

$$8u_1 + 2u_2 + u_3 - 3u_5 = (r, 2 - r),$$
$$8u_1 + 2u_2 + u_3 - 3u_5 = (r, 2 - r),$$
$$-2u_1 + 5u_2 + u_3 - u_4 + u_5 = (4 + r, 7 - 2r), \quad (20)$$
$$u_1 - u_2 + 5u_3 + u_4 + u_5 = (1 + 2r, 6 - 3r),$$
$$-u_3 + 4u_4 + 2u_5 = (1 + r, 3 - r),$$
$$u_1 - 2u_2 + 3u_5 = (3r, 6 - 3r).$$

Then we have the system as in equation (21) given below.

Figure 6 presents a comparison between the proposed approach solution in figure 6(a) and the exact solution (22) obtained by Dehghan & Hashemi (2006) in figure 6(b) with $0 \le r \le 1$.

$$u_1 = (\underline{u}_1, \overline{u_1}) = (.7287 - .33057r, \quad .044135 + .35399r),$$
$$u_2 = (\underline{u}_2, \overline{u_2}) = (.61418 + .16626r, \quad 1.0773 - .29682r),$$
$$u_3 = (\underline{u}_3, \overline{u_3}) = (.12628 + .29059r, \quad .91822 - .50136r),$$
$$u_4 = (\underline{u}_4, \overline{u_4}) = (.24192 - .33149r, -.4158 + .32622r),$$
$$u_5 = (\underline{u}_5, \overline{u_5}) = (.47528 + .91231r, 2.3947 - 1.0072r). \quad (22)$$

The objective error (9) and the numerical error for the proposed approach preview in table 3. The error for the variables u_i with $0 \le r \le 1$ is previewed in figure 7.

$$\begin{bmatrix} f_1\\ f_2\\ f_3\\ f_4\\ f_5\\ f_6\\ f_7\\ f_8\\ f_9\\ f_{10} \end{bmatrix} = \begin{bmatrix} 8&2&1&0&0&0&0&0&0&3\\ 0&5&1&0&1&2&0&0&1&0\\ 1&0&5&1&1&0&1&0&0&0\\ 0&0&0&4&2&0&0&1&0&0\\ 1&0&0&0&3&0&2&0&0&0\\ 0&0&0&0&3&8&2&1&0&0\\ 2&0&0&1&0&0&5&1&0&1\\ 0&1&0&0&0&1&0&5&1&1\\ 0&0&1&0&0&0&0&0&4&2\\ 0&2&0&0&0&1&0&0&0&3 \end{bmatrix} \begin{bmatrix} \underline{u}_1\\ \underline{u}_2\\ \underline{u}_3\\ \underline{u}_4\\ \underline{u}_5\\ -\overline{u_1}\\ -\overline{u_2}\\ -\overline{u_3}\\ -\overline{u_4}\\ -\overline{u_5} \end{bmatrix} - \begin{bmatrix} r\\ 4+r\\ 1+2r\\ 1+r\\ 3r\\ 2-r\\ 7-2r\\ 6-3r\\ 3-r\\ 6-3r \end{bmatrix}. \qquad (21)$$

n	p	Objective error	$\|\|.\|\|_2$
5	10	2.1940e-28	3.2204e-14

Table 3: Execution data for example 6.3 with $r = 1$

Table 4: Present a comparison between the obtained error from the proposed approach and the previewed error in Dehghan & Hashemi (2006). It is clear the difference between the proposed approach error and the other methods error.

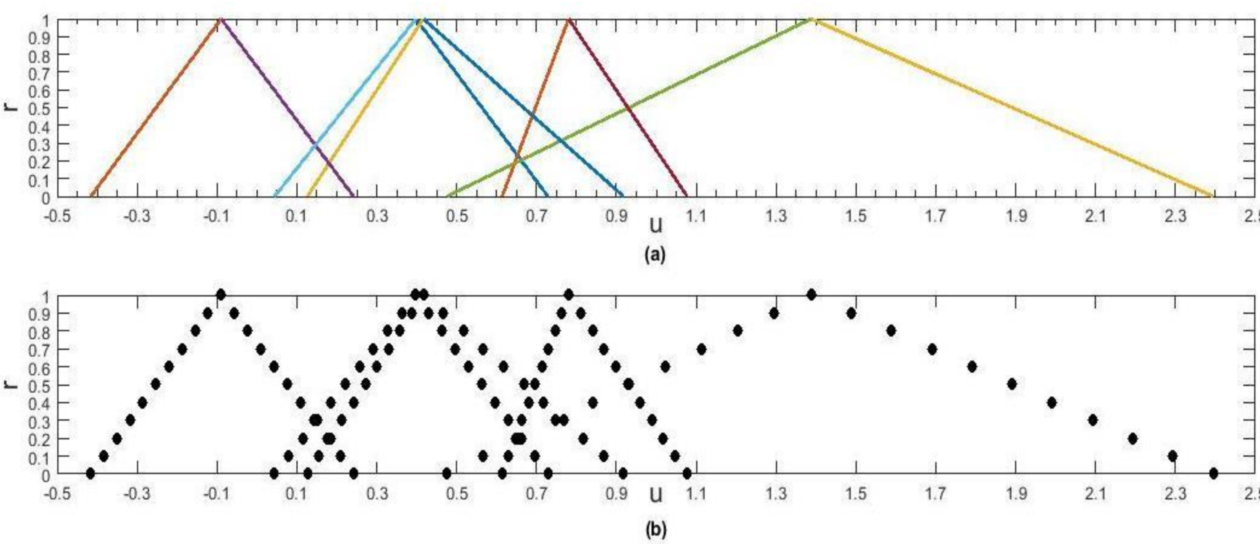

Figure 6: (a) Approximate solution for example 6.3. by the introduced approch, (b) The exact solution of equation (22).

Method	The proposed approach	Jacobi	Forward Guass-seidel	Backward Guass-seidel	EGS	Forward SOR	Backward SOR	JOR
Error	3.2204e-14	0.0045	0.0034	0.0014	0.0033	0.0012	0.0035	0.0027

Table 4: Comparison between errors in different methods.

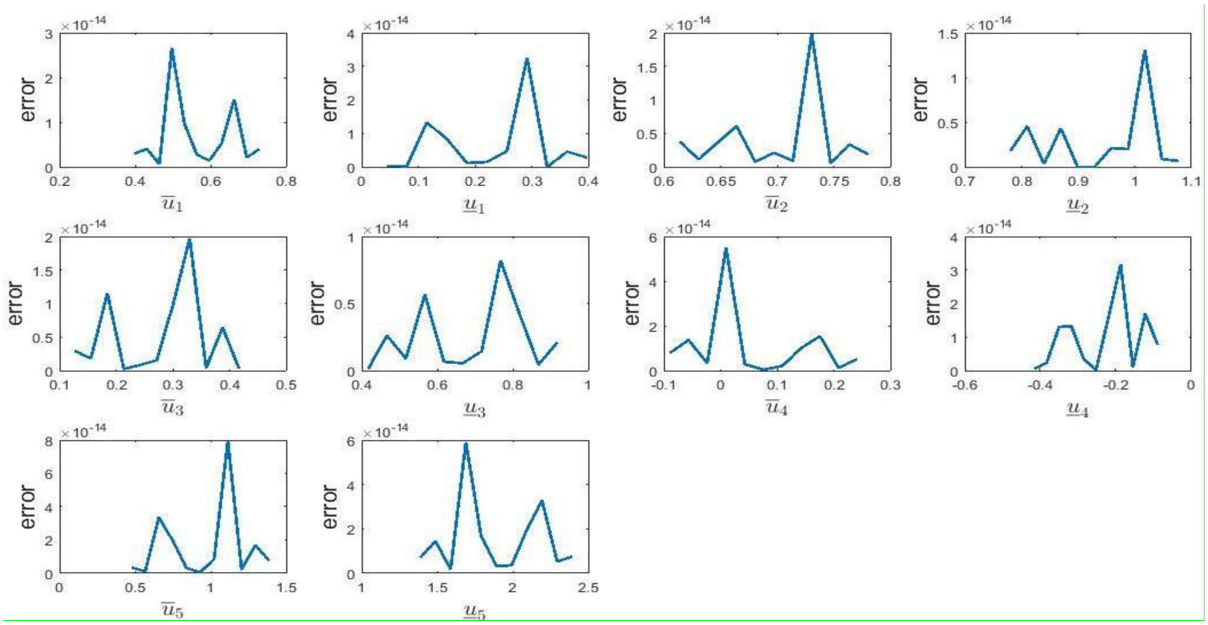

Figure 7: The error for the variables u_i for example 6.3.

Example 6.4. Studying this one of the most important examples,

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ . & & & & & & & & \\ . & & & & & & & & \\ . & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ . \\ . \\ . \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} (-5+3r, 1-3r) \\ (-4+4r, 4-4r) \\ . \\ . \\ . \\ (-4+4r, 4-4r) \\ (-5+3r, 1-3r) \end{bmatrix}. \tag{23}$$

Similarly, we have:

$$\begin{bmatrix} f_1 \\ f_2 \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ f_{p-1} \\ f_p \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0\ldots\ldots0 & 0 & 0 & 2 & 0 & 0\ldots\ldots0 & 0 & 0 \\ 1 & 0 & 1\ldots\ldots0 & 0 & 0 & 0 & 2 & 0\ldots\ldots0 & 0 & 0 \\ . & & & & & & & & & \\ . & & & & & & & & & \\ 0 & 0 & 0\ldots1 & 0 & 1 & 0 & 0 & 0\ldots0 & 2 & 0 \\ 0 & 0 & 0\ldots0 & 1 & 0 & 0 & 0 & 0\ldots0 & 0 & 2 \\ 2 & 0 & 0\ldots0 & 0 & 0 & 0 & 1 & 0\ldots0 & 0 & 0 \\ 0 & 2 & 0\ldots0 & 0 & 0 & 1 & 0 & 1\ldots0 & 0 & 0 \\ . & & & & & & & & & \\ . & & & & & & & & & \\ 0 & 0 & 0\ldots0 & 2 & 0 & 0 & 0 & 0\ldots1 & 0 & 1 \\ 0 & 0 & 0\ldots0 & 0 & 2 & 0 & 0 & 0\ldots0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \\ . \\ . \\ \underline{u}_{n-1} \\ \underline{u}_n \\ -\overline{u}_1 \\ -\overline{u}_2 \\ . \\ . \\ -\overline{u}_{n-1} \\ -\overline{u}_n \end{bmatrix} - \begin{bmatrix} -5+3r \\ -4+4r \\ . \\ . \\ -4+4r \\ -5+3r \\ -1+3r \\ -4+4r \\ . \\ . \\ -4+4r \\ -1+3r \end{bmatrix}. \tag{24}$$

The results of this example are introduced in figure 8, figure 9 and table 5. When the number of the variables is 20, we present in figure 8 the approximate solutions compared with the exact solutions $u_i = (1 + r, 3 - r)$, for $i = 1,2,\ldots,20$. Also figure 9 preview the execution time with the number of objective functions p after interpolation the obtained data by cubic spline. Table 5 presents the numerical error for the proposed approach and the objective error (9) for varies numbers of fuzzy variable when $r = .5$.

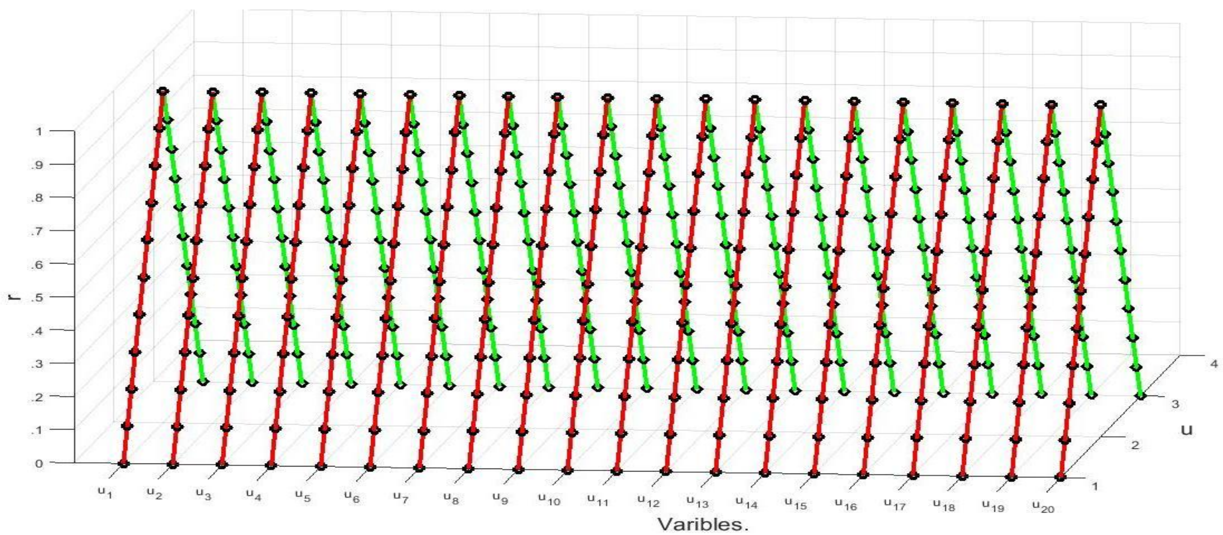

Figure 8: Exact and approximate solutions (circles for $\underline{u}_i$ and diamond for $\overline{u_i}$) for example 6.4.

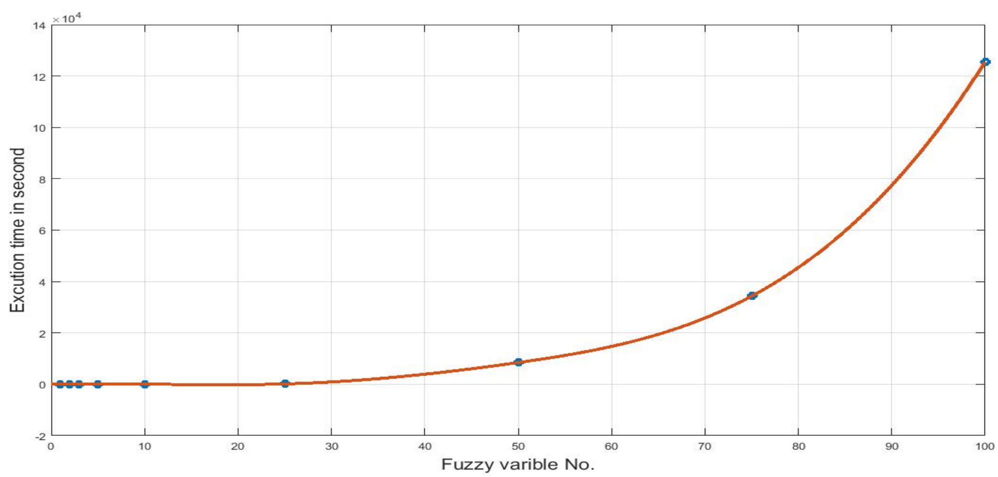

Figure 9: CPU time and the number of objective functions p (circles for the obtained data and the line for interpolated data) for example 6.4.

Remark 6.4. Applying the well-known methods Jacobean and Gauss-Seidel for solving the general case of our study fails. Because these methods must divide over the diagonal elements of the matrix D in equation (8), that means the diagonal elements d_{ll} must be, > 0 and this is not satisfied in all problems. Lubna Inearat and Naji Qatanani in the paper Inearat & Atanani (2018) depended on Jacobean and Gauss-Seidel methods for solving small systems, but all solved examples have values > 0 in the diagonal elements. Even if there are classical iterative methods that can treat this problem, or one can manipulate the system of equations to attain the requirement of the classical methods. Still, the large scale of the problem is a big handicap, and the numerical method will not diverge when n increases.

n	p	Objective error	$\|\|.\|\|_2$
1	2	7.36861E-27	2.42848E-13
2	4	2.90005E-28	2.41376E-14
3	6	9.37173E-27	5.01252E-14
5	10	1.19294E-27	3.33524E-14
10	20	4.77772E-25	6.08422E-14
25	50	1.4569E-25	1.92699E-14
50	100	4.47527E-24	2.52075E-14
75	150	8.12074E-22	2.18041E-13
100	200	1.19521E-22	7.26635E-14

Table 5: Execution data for example 6.3 with $r = .5$

Conclusion

This paper introduced a modified approach to solve one of the most important problems. The new approach solves the large scale of a fuzzy linear system by transforming the system into a multi-objective optimization problem. Then the multi-objective system is converted to a single optimization problem by the weight sum method. Our strategy is to solve the resulting single-objective problem numerically with high accuracy. We proved that the multi-objective functions resulting from the fuzzy linear system are convex. Applying the LFA for the resulting convex function makes the iterated approximations tend to be the optimal solution. The previewed results for some examples are satisfied with a small acceptable error. Also, the CPU time is small enough with increasing the number of fuzzy variables. A comparison is made between the proposed approach and other approaches to ensure the accuracy of the approach. The proposed approach can be extended to deal with more complicated stochastic systems.

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