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Research Article:

Bell Polynomials for Beltrami Operators

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Abstract

The Bell polynomials, which known in name of Eric Temple Bell, are widely used in combinatorial mathematics to explore set partitions. The Bell polynomials have a relation to Bell numbers and Stirling numbers. They can also be found in a various applications, like the Faà di Bruno formula. In this paper, we define generalized Bell polynomials, and investigate basic properties of these polynomials. We find Bell polynomials for the Beltrami operator. Also, we obtain explicit formulas for the powers of the Beltrami operator and a generalized Beltrami operator. We introduce an application of the obtained Bell polynomial for the Beltrami operator, namely, a Beltrami- Faà di Bruno formula is established. Illustrative examples are given.

1. Introduction

Bell polynomials with partial multivariate introduced by E.T. Bell (Bell, 1934; Aboud et al., 2017). However, the credit for their name comes backs to the work of Riordan (Riordan, 2014), who investigated the Faa' di Bruno formula (Faa' di Bruno, 1855; Faa' di Bruno, 1857) and discovered how to express the higher order derivative of a composition fog in terms of the derivatives of both f and g.

Bell polynomials $B_n = B_n(t_1, t_2, \dots t_n)$, $n = 0, 1, 2, \dots$, have so many uses in combinatorics, analysis, algebra, probability theory, and other fields. Just a few fundamental examples will suffice:

1) In probability theory, the n^{th} moment of a probability distribution is a full Bell polynomial of the cumulants.
2) Lagrange inversion and Bell polynomials are related. The Faa' di Bruno formula leads to this.
3) In various combinatorial formulae for the Bell polynomials, Lah numbers, Stirling numbers, and other combinatorial numbers are involved.

The Faa' di Bruno formula and several related combinatorial identities have been pre-

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sented in (Comtet, 2012). Many applications and formulae of the Bell polynomials have been introduced in the work of Mihoubi (Mihoubi, 2008). In (Comtet, 2012)

$$B_{n,k}(t_1,t,\dots,t_{n-k+1}) = \sum_{\substack{1\le i\le n-k+1\\ \ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n-k+1} i\ell_i=n\\ \sum_{i=1}^{n-k+1}\ell_i=k}} \frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{t_i}{i!}\right)^{\ell_i}. \tag{1.1}$$

The Faa' di Bruno formula can be described in terms of the Bell polynomials of the second kind $B_{n,k}(t_1,t,\dots,t_{n-k+1})$ by, see (Feng et al., 2017)

$$\frac{d^n}{dt^n} f\circ g(t) = \sum_{j=0}^{n} f^{(j)}(g(t))B_{n,j}\left(g'(t),\dots,g^{(n-j+1)}(t)\right). \tag{1.2}$$

By (1.1) we can easily deduce that, for $n\ge j\ge 0$,

$$B_{n,j}(1,0,\dots,0) = B_{n,j}\left(\frac{d}{dt}t,\frac{d^2}{dt^2}t,\dots,\frac{d^{n-j+1}}{dt^{n-j+1}}t\right) = \binom{0}{n-j} = \begin{cases}1, & n=j,\\ 0, & n\ne j.\end{cases} \tag{1.3}$$

In (Feng et al., 2017; Feng et al., 2020a), it was established that the second kind Bell polynomials $B_{n,j}(t_1,t_2,\dots,t_{n-k+1})$ satisfy

$$B_{n,j}(t,1,0,\dots,0) = \frac{1}{2^{n-j}}\frac{n!}{j!}\binom{j}{n-j}t^{2j-n},\ 0\le j\le n, \tag{1.4}$$

where $\binom{0}{0}=1$, and $\binom{r}{s}=0$ for $0\le r<s$. Since

$$B_{n,j}(\alpha\beta t_1,\alpha\beta^2 t_2,\dots,\alpha\beta^{n-j+1}t_{n-j+1}) = \alpha^j\beta^n B_{n,j}(t_1,t_2,\dots,t_{n-j+1}),\qquad n\ge j\ge 0, \tag{1.5}$$

we can rewrite (1.4) as, see (Comtet, 2012)

$$B_{n,j}\left(\frac{d}{dt}t^2,\frac{d^2}{dt^2}t^2,\dots,\frac{d^{n-j+1}}{dt^{n-j+1}}t^2\right) = B_{n,j}(2t,2,0,\dots,0) = 2^j B_{n,j}(t,1,0,\dots,0) = \frac{n!}{j!}\binom{j}{n-j}(2t)^{2j-n}, \tag{1.6}$$

for $n\ge j\ge 0$. Thus, combining (1.2) with (1.6), helps in computing the higher order derivative for functions of the type $f(at^2+bt+c)$ see (Feng et al., 2020a), such as

$$e^{\pm t^2}, \sin(t^2),\ \cos(t^2), \ln(1\pm t^2),$$
$$\ln(1\pm t^2)^{\alpha}, \arcsin t, \arccos t, \arctan t.$$

In (Feng el al., 2019; Edition et al., 2019), the following formulas are obtained

$$B_{2n,j}(0,2!,\dots,0,(2n)!) = \frac{(2n)!}{j!}\binom{n-1}{j-1},$$
$$B_{2n-1,j}(0,2!,\dots,0,(2j-2)!,0) = 0,$$
$$B_{2n,2j}(1!,0,\dots,(2n-1)!,0) = \frac{(2n)!}{(2j)\,!}\binom{n+j-1}{2j-1},$$
$$B_{2n,2j-1}(1!,0,\dots,(2n-1)!,0) = 0,$$
$$B_{2n-1,2j-1}(1!,0,\dots,(2n-1)!,0) = \frac{(2n-1)!}{(2j-1)\,!}\binom{n+j-2}{2j-2},$$
$$B_{2n-1,2j}(1!,0,\dots,0,(2n-1)!) = 0,$$

For simplicity, we denote

$$\lambda(n,j) = B_{n,j}\left(0,2!,0,4!,\dots,(n-j+1)!\frac{1+(-1)^{n-j+1}}{2}\right)$$

and

$$\mu(n,j) = B_{n,j}\left(1!,0,3!,0,\dots,(n-j+1)!\ \frac{1+(-1)^{n-j+1}}{2}\right).$$

Combining these results, the above claims can be rewrite as

$$\lambda(2n,j) = \frac{(2n)!}{j\,!}\binom{n-1}{j-1},$$
$$\lambda(2n-1,j) = 0,$$
$$\mu(2n,2j) = \frac{(2n)!}{(2j)!}\binom{r+j-1}{2j-1},$$
$$\mu(2n,2j-1) = 0,$$
$$\mu(2n,2j-1) = \frac{(2n-1)!}{(2j-1)!}\binom{n+j-2}{2j-2},$$
$$\mu(2n-1,2j) = 0.$$

In (Feng et al., 2020b) the authors surveyed many formulae and applications of the polynomials $B_{n,j}(t_1,t_2,\dots,t_{n-j+1})$, in which $t_1,t_2,\dots,t_{n-j+1}$ were replaced by some elementary functions. Here, we restate some of these applications:

1) Exponential function. The n^{th} derivative for a function of the type $f(e^t)$ like as $\frac{1}{e^{\pm t}\pm 1}$,

can be found by the Faa' di Bruno formula (1.2). It needs to compute

$$B_{n,j}\big((e^{\pm t})', (e^{\pm t})'', \ldots, (e^{\pm t})^{(n-j+1)}\big)$$
$$= B_{n,j}(\pm e^{\pm t}, (\pm 1)^2 e^{\pm t}, \ldots, (\pm 1)^{n-j+1} e^{\pm t})$$
$$= (\pm 1)^n e^{\pm jt} B_{n,j}(1, 1, \ldots, 1).$$

In (Feng et al., 2020b), the following result is listed

$$S(n,j) = B_{n,j}(1, 1, \ldots, 1),$$

where $S(n,j)$ points to the second kind Stirling numbers. Thus, we can easily obtain

$$\left(\frac{1}{e^{\pm t} \pm 1}\right)^{(n)} = (\pm 1)^n \sum_{\nu=0}^{n} (-1)^\nu \nu!\, S(n,\nu) \frac{e^{\pm \nu t}}{(e^{\pm t} \pm 1)^{\nu+1}}.$$

2) Logarithmic function. The n^{th} derivative of the composition $f(\ln(t+1))$, for example $\frac{1}{\ln(t+1)}$, can be obtained by using identities (1.2) and (1.5). Here, one needs to compute

$$B_{n,j}\big([\ln(1+t)]', [\ln(1+t)]'', \ldots, [\ln(1+t)]^{(n-j+1)}\big)$$
$$= \frac{(-1)^{n-j}}{(1+t)^n} B_{n,j}(0!, 1!, \ldots, (n-j)!).$$

In (Feng et al 2020b), the following result is listed

$$B_{n,j}(0!, 1!, \ldots, (n-j)!) = (-1)^{n-j} s(n,j),$$

where $s(n,j)$ denotes the first kind Stirling numbers. Hence, we have

$$\left[\frac{1}{\ln(1+t)}\right]^{(n)} = \frac{1}{(1+t)^n} \sum_{\nu=0}^{n} \frac{(-1)^\nu \nu!\, s(n,\nu)}{\ln^{\nu+1}(1+t)}.$$

Further, we recall some facts on the Bell polynomials. The recursion equation (Schimming & Rida, 1996)

$$B_0 = 1,\ \ B_{n+1} = \sum_{\nu=0}^{n} \binom{n}{\nu} B_{n-\nu} t_{\nu+1},\ \ n \geq 1. \quad (1.7)$$

And the generating function of the Bell polynomials is, see (Feng et al., 2019)

$$\sum_{n=1}^{\infty} \frac{B_n}{n!} x^n = \exp \sum_{n=1}^{\infty} \frac{t_n}{n!} x^n \quad (1.8)$$

Bell polynomials $B_n = B_n(t_1, t_2, \ldots, t_n)$ can be expressed explicitly as, see (Feng et al., 2019; Feng et al., 2020a; Kaufmann, 1968; Rida, 1996)

$$B_n(t_1, t_2, \ldots, t_n)$$
$$= \sum_{\|j\|=n} \frac{n!}{j!} \left(\frac{t_1}{1!}\right)^{j_1} \left(\frac{t_2}{2!}\right)^{j_2} \cdots \left(\frac{t_n}{n!}\right)^{j_n}, \quad (1.9)$$

where $j_1 \geq 0, j_2 \geq 0, \ldots, j_n \geq 0$, $j! = j_1! j_2! \ldots j_n!$, and $\|j\| = j_1 + 2j_2 + \cdots + nj_n$.

Further, the bell polynomials can be decomposed into its homogeneous parts as. See (Schimming & Rida, 1996)

$$B_n(t_1, t_2, \ldots, t_n) = \sum_{j=1}^{n} B_{n,j}(t_1, t_2, \ldots, t_n), \quad (1.10)$$

with

$$B_{n,j}(t_1, t_2, \ldots, t_n)$$
$$= \sum_{\substack{|j|=k \\ \|j\|=n}} \frac{n!}{j!} \left(\frac{t_1}{1!}\right)^{j_1} \left(\frac{t_2}{2!}\right)^{j_2} \cdots \left(\frac{t_n}{n!}\right)^{j_n}, \quad (1.11)$$

where $j = (j_1, j_2, \cdots, j_n)$, $|j| = j_1 + j_2 + \cdots + j_n$, and $\|j\| = j_1 + 2j_2 + \cdots + nj_n$, is called k-homogeneous Bell polynomials, that is,

$$B_{n,k}(\lambda t_1, \lambda t_2, \ldots, \lambda t_n) = \lambda^k B_{n,k}(t_1, t_2, \ldots, t_n). \quad (1.12)$$

The d-homogenous polynomials $B_{n,k}$ appear in the iterated chain rule of Faa di Bruno formula, see (Faa'di Bruno, 1855; Faa'di Bruno, 1857; Schimming & Rida, 1996; Kaufmann, 1968; Rida, 1996).

Furthermore, Bell polynomials help find explicit formulas for many differential operators and thus motivate to study of higher-order differential equations. The Beltrami operator is one of the well-known complex differential operators, whose higher-order boundary value problems have not received adequate attention despite the importance of these problems. Therefore, we are motivated to find explicit formulas for the powers of Beltrami operators in terms of Bell polynomials. Then, we use the results of this study to solve boundary value problems for higher-order differential operators including powers of Beltrami operators as main parts.

The manuscript is structured as follows. In Section 2, we find the Bell polynomials for the Beltrami operator. In Section 3, we study an explicit formula for powers of the Beltrami and the powers of a generalized Beltrami operator. An alternative induction proof for this explicit formula is given Section 4. Section 5 is devoted to introduce an application of the Bell polynomial for the Beltrami operator. We establish a Beltrami- Faa di Bruno formula. Illustrative examples are given. Finally, some concluding remarks are given.

2. The Bell polynomials for the Beltrami operator

The importance of the Beltrami operator follows from the fact that the theory of Beltrami equations is related with various problems in analysis and geometry. For more information one can refer to (Bojarski, 1988; Bojarski et al., 2013; Gutlyanskii et al., 2012; Katz et al., 2018; Pastukhova, 2017). Next, we establish the Bell polynomial for the Beltrami operator.

Theorem 1. Let $l=\rho\partial_{\bar{z}}+q\partial_z$ be the Beltrami operator. For a function $\omega=\omega(z)$ there holds

$$e^{-\boldsymbol{\omega}(\boldsymbol{z})}l^n e^{\boldsymbol{\omega}} = B_{n,l}(\omega_z,\omega_{\bar{z}},,\dots,\omega_{z^n},\omega_{z^{n-1}\bar{z}},\dots,\omega_{\bar{z}^n}), \tag{2.1}$$

where $B_{n,l}$ is the Bell polynomial for the Beltrami operator.

Proof. Let $B_n^l=e^{-\boldsymbol{\omega}(\boldsymbol{z})}l^n e^{\boldsymbol{\omega}(\boldsymbol{z})}$. We have to show that $\mathrm{B}_{\mathrm{n}}^l$ are the required Bell polynomials. For $n\geq 1$ we have

$$B_{n+1}^l=e^{-\boldsymbol{\omega}(\boldsymbol{z})}l^{n+1}e^{\boldsymbol{\omega}(\boldsymbol{z})}=e^{-\boldsymbol{\omega}(\boldsymbol{z})}l^n[l(e^{\boldsymbol{\omega}(\boldsymbol{z})})]$$

$$=e^{-\boldsymbol{\omega}(\boldsymbol{z})}l^n[e^{\boldsymbol{\omega}}l\omega]$$

$$=e^{-\boldsymbol{\omega}(\boldsymbol{z})}\sum_{k=0}^{n}\binom{n}{k}l^k[(l\omega)]l^{n-k}\left[e^{\omega(z)}\right]$$

$$=e^{-\boldsymbol{\omega}(\boldsymbol{z})}\sum_{k=0}^{n}\binom{n}{k}[(l^{k+1}\omega)]\left(l^{n-k}e^{\omega(z)}\right)$$

$$=\sum_{k=0}^{n}\binom{n}{k}[(l^{k+1}\omega)]\left(e^{-\omega}l^{n-k}e^{\omega(z)}\right)$$

$$=\sum_{k=0}^{n}\binom{n}{k}B_{n-k}^l[(l^{k+1}\omega)], \tag{2.2}$$

formally, we set

$$B_0^l=1. \tag{2.2$'$}$$

This show that $B_n^l=e^{-\boldsymbol{\omega}(\boldsymbol{z})}l^n e^{\boldsymbol{\omega}(\boldsymbol{z})}$ are the Bell polynomials. This completes the proof.

The non-commutative sequence

$$B_n^l=e^{-\boldsymbol{\omega}(\boldsymbol{z})}l^n e^{\boldsymbol{\omega}(\boldsymbol{z})}, n=1,2,\dots$$

begins with

$$B_0^l=1,\qquad B_1^l=l\,\omega,\quad B_2^l=l^2\omega+(l\omega)^2,$$
$$B_3^l=l^3\omega+3l\omega l^2\omega+2l\omega l^2\omega+(l\omega)^3,$$
$$B_4^l=l^4\omega+4l\omega l^3\omega+3\,(l\omega)^2.\,l^2\omega+3(l^2\omega)^2 + 3\,l\omega\,l^2\omega+(l\omega)^4.$$

For more explicit

$$B_1^l=\rho\omega_{\bar{z}}+q\omega_z$$
$$B_2^l=\rho^2(\omega_{\bar{z}})^2+\rho^2\omega_{\bar{z}\bar{z}}+\rho q_{\bar{z}}\omega_z+2\rho q\omega_z\omega_{\bar{z}} +2\rho q\omega_{z\bar{z}}+qq_z\omega_z+q^2(\omega_z)^2+q^2\omega_{zz}$$

Let us apply multi-index formalism in order to present an explicit expression for the Bell polynomials. We collect integers $j\geq 0, j_2\geq 0,\dots,j_n\geq 0$ to a multi-index $j=(j_1,j_2,\dots,j_n)$, and we set

$$j!=j_1!\,j_2!\dots j_n!\,,\qquad |j|=j_1+j_2+\dots+j_n\,,$$
$$\|j\|=j_1+2j_2+\dots+nj_n$$

We have to start a technical result for formal powers series

Lemma 1. (Rida, 1996) If $\alpha_{m0}=1$ for $m=1,2,\cdots$, then

$$\prod_{m=1}^{\infty}\Big(\sum_{n=0}^{\infty}\alpha_{mn}x^{mn}\Big)=\sum_{n=0}^{\infty}\mathrm{A}_n x^n, \tag{2.3}$$

where

$$\mathrm{A}_0=1,\mathrm{A}_n=\sum_{\|j\|=n}\alpha_{1j_1}\alpha_{2j_2}\dots\alpha_{nj_n}\,,n\ \geq 1, \tag{2.4}$$

that means, the sum in (2.4) runs through all non-negative integers $j=(j_1,j_2,\dots,j_n)$ such that $\|j\|=j_1+2j_2+\dots+nj_n=n$.

Thus, we can define a sequence of Bell polynomials.

Definition 1. The sequence of Bell polynomials

$$B_0=1,B_n=B_n\left(\omega_1,\omega_2,\dots,\omega_{\frac{n(n+3)}{2}}\right),n=1,2,\dots,$$

in the (finitely many) variables $\omega_1, \omega_2, \dots, \omega_{\frac{n(n+3)}{2}}$, is defined through a generating function

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = \exp\left(\sum_{n=1}^{\infty} \frac{\omega_n}{n!} x^n\right).$$

Consequentially, an explicit expression for the Bell polynomial can be established.

Theorem 2. The n^{th} Bell polynomial is explicitly given by

$$B_0 = 1, B_n \left(\omega_1, \omega_2, \dots, \omega_{\frac{n(n+3)}{2}}\right)$$
$$= \sum_{\|k\|=n} \frac{n!}{k!} \left(\frac{\omega_1}{1!}\right)^{k_1} \left(\frac{\omega_2}{2!}\right)^{k_2} \dots \left(\frac{\omega_{\frac{n(n+3)}{2}}}{n!}\right)^{k_n} \text{ for } n \geq 1.$$
$$B_0 = 1.$$

For $m = \frac{n(n+3)}{2}$, $m \geq 1$ then

$$B_m \ (\omega_1, \omega_2, \dots, \omega_m)$$
$$= \sum_{\|k\|=m} \frac{m!}{k!} \left(\frac{\omega_1}{1!}\right)^{k_1} \left(\frac{\omega_2}{2!}\right)^{k_2} \dots \left(\frac{\omega_m}{m!}\right)^{k_m}$$

Proof. We apply the functional relation in Lemma 1

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = \exp\left(\sum_{m=1}^{\infty} \frac{\omega_m}{m!} x^m\right) = \prod_{m=1}^{\infty} \exp\left(\frac{\omega_m}{m!} x^m\right)$$
$$= \prod_{m=1}^{\infty} \left(\sum_{n=0}^{\infty} \alpha_{mn} x^{mn}\right) = \sum_{n=0}^{\infty} \mathrm{A}_n x^n$$

where,

$$\alpha_{mn} = \frac{1}{n!} \left(\frac{\omega_m}{m!}\right)^n.$$

Hence, with $m = \frac{n(n+3)}{2}$

$$\mathrm{A}_m = \sum_{\|j\|=m} \alpha_{1j_1} \alpha_{2j_2} \dots \alpha_{mj_m}$$
$$= \sum_{\|j\|=m} \frac{1}{j_1! j_2! \dots j_m!} \left(\frac{\omega_1}{1!}\right)^{j_1} \left(\frac{\omega_2}{2!}\right)^{j_2} \dots \left(\frac{\omega_m}{m!}\right)^{j_m}$$
$$= \sum_{\|j\|=m} \frac{1}{j!} \left(\frac{\omega_1}{1!}\right)^{j_1} \left(\frac{\omega_2}{2!}\right)^{j_2} \dots \left(\frac{\omega_m}{m!}\right)^{j_m}.$$

Comparison of the coefficients of the powers of z gives the result.

Now, we are going to establish the Bell polynomial for the Beltrami operator. Let $\omega = \varphi(u)$, $u = u(z, \bar{z})$.The $l^n\omega$ for all $n \geq 1$ can be obtained in terms of the derivatives of u, that are $lu, l^2u, \dots, l^nu$ as follows.

$$l\omega = \omega_{\bar{z}} + q\omega_z = \varphi' u_{\bar{z}} + q\varphi' u_z = \varphi'. lu,$$
$$l^2\omega = l[\omega_{\bar{z}} + q\omega_z] = \partial_{\bar{z}}[\varphi'. lu] + q\partial_z[\varphi'. lu]$$
$$= \varphi''(lu)^2 + \varphi' l^2 u,$$
$$l^3\omega = \varphi'''(lu)^2. lu + 2\varphi''(lu). (l^2u)$$
$$+ \varphi'' l^2u. lu + \varphi' l^3 u$$
$$= \varphi'''(lu)^3 + 3\varphi''(lu)(l^2u) + \varphi'^{l^3} u,$$
$$l^4\omega = \varphi''''. lu + 3\varphi'''(lu). (l^2u)$$
$$+3\varphi''' lu. l^2u. lu\ 3\varphi'' l^2u. l^2u$$
$$+3\varphi'' lu. l^3u + \varphi''\ \ l^3u. lu + \varphi'^{\,l^4} u,$$
$$= \varphi''''(lu)^4 + 3\varphi'''(lu). (l^2u)$$
$$+3\varphi'''(lu)^2. l^2u\ 3\varphi''(l^2u)^2$$
$$+4\varphi'' lu. l^3u\varphi'\ l^4u.$$

Continuing this process, we obtain the following iterated chain rule

$$l^n\omega = \sum_{k=1}^{n} \varphi^{(d)}(u) B_{n,k}(lu, l^2u, \dots, l^nu).$$

The following theorem gives an explicit formula for $B_{n,k}(lu, l^2u, \dots, l^nu)$.

Theorem 3. There holds

$$B_{n,j} = \sum_{n_2,\dots,n_j=1}^{n} \binom{n-1}{n_2}\binom{n_2-1}{n_3} \dots \binom{n_{j-1}-1}{n_j} \times$$
$$\times\, l^{n_j}\omega\ l^{n_{j-1}-n_j}\ \omega \dots l^{n_2-n_3}\ \omega\ l^{n-n_2}\ \omega$$

for $n \geq j \geq 2$ and $B_0 = 1, B_1 = l\omega$.

Proof. Let us rewrite (2.2) in the form

$$B_n = l^n\omega + \sum_{j_1=1}^{n-1} \binom{n-1}{j_1} B_{j_1}[l^{n-j_1}\omega].$$

Here we insert

$$B_{j_1} = l^{j_1}\omega + \sum_{j_2=1}^{j_1-1} \binom{j_1-1}{j_2} B_{j_2}[l^{j_1-j_2}\omega],$$

$$B_n = l^n\omega + \sum_{j_1=1}^{n-1} \binom{n-1}{j_1} l^{j_1}\omega l^{n-j_1}\omega$$

$$+\sum_{j_1=1}^{n-1}\sum_{j_2=1}^{j_1-1}\binom{n-1}{j_1}\binom{j_1-1}{j_2} B_{j_2}[l^{j_1-j_2}\omega][l^{n-j_1}\omega].$$

$$B_{j_2}=l^{j_2}\omega+\sum_{j_3=1}^{j_2-1}\binom{j_2-1}{j_3} B_{j_3}[l^{j_2-j_3}\omega].$$

Then,

$$B_n=l^n\omega+\sum_{j_1=1}^{n-1}\binom{n-1}{j_1} l^{j_1}\omega l^{n-j_1}\omega$$
$$+\sum_{j_1=1}^{n-1}\sum_{j_2=1}^{j_1-1}\sum_{j_3=1}^{j_2-1}\binom{n-1}{j_1}\binom{j_1-1}{j_2}\binom{j_2-1}{j_3}$$
$$\times B_{j_3}[l^{j_2-j_3}\omega][l^{j_1-j_2}\omega l^{n-j_1}\omega]$$

After the k_d-step there are summation indices $j_1, j_2, \dots, j_k$ such that

$$n>j_1>j_2>\cdots>j_k\geq 1,$$

then we get,

$$B_n=l^n\omega+\sum_{j_1=1}^{n-1}\binom{n-1}{j_1} l^{j_1}\omega l^{n-j_1}\omega$$
$$+\sum_{j_1=1}^{n-1}\sum_{j_2=1}^{j_1-1}\binom{n-1}{j_1}\binom{j_1-1}{j_2} l^{j_2}\omega\, l^{j_1-j_2}\omega\, l^{n-j_1}\omega+\cdots$$
$$+\sum_{j_1=1}^{n-1}\sum_{j_2=1}^{j_1-1}\cdots\sum_{j_k=1}^{j_{k-1}-1}\binom{n-1}{j_1}\binom{j_1-1}{j_2}\cdots\binom{j_{k-1}-1}{j_k}$$
$$\times\ l^{j_k}\omega\, l^{j_{k-1}-j_k}\omega \dots\ l^{n-j_1}\omega$$
$$+\sum_{j_1=1}^{n-1}\sum_{j_2=1}^{j_1-1}\cdots\sum_{j_{k+1}=1}^{j_k-1}\binom{n-1}{j_1}\binom{j_1-1}{j_2}\cdots\binom{j_k-1}{j_{k+1}}$$
$$\times B_{j_{k+1}}[l^{j_{k+1}-j_k}\omega]\dots[l^{j_1-j_2}\omega l^{n-j_1}\omega] \qquad (2.5),$$

The process stop at $j=n-1$ where

$$j_1=n-1, j_2=n-2, \dots, j_{n-1}=1,$$

and $B_1=\rho\omega_{\bar{z}}+q\omega_z=l\omega$.

The expression for B_n appears properly decomposed into its homogeneous parts $B_{n,k}$ (Feng et al., 2019; Feng et al., 2020; Schimming & Rida, 1996).

In the final result, we rename the summation indices and let the formally run through $1,2,\dots,n$. Actually, we have in (1.9) $n>j_1>j_2>\cdots>j_{k-1}\geq 1$,

Since for other values at least one binomial coefficient vanishes. If $j=2$ then we interpret $j_1=n$.

In case of noncommutativity, it makes a difference when we replace (2.2), (2.2′) by

$$B_0^*=1,\ \ B_{n+1}^*:=\sum_{j=0}^{n}\binom{n}{j}\left[l^{j+1}\omega\right]B_{n-j}^*.$$

Let us call the unique solutions

$$B_n^*=B_n^*\left(\omega_1,\omega_2,\dots,\omega_{\frac{n(n+3)}{2}}\right), (n=0,1,2,\dots)$$

the dual (noncommutative) Bell polynomials. It is easy to see that

$$B_n^*:=\sum_{k=1}^{n}B_{n,k}^*, \qquad (2.6)$$

with

$$B_{n,k}^*=\sum_{n_2,\dots,n_k=1}^{n}\binom{n-1}{n_2}\binom{n_2-1}{n_3}\dots\binom{n_{k-1}-1}{n_k}\times$$
$$\times\ l^{n-n_2}\ \omega l^{n_2-n_3}\ \omega\ \dots l^{n_{k-1}-n_k}\omega l^{n_k}\omega\ .$$

3. Formulas of powers of a generalized Beltrami operator

3.1. An explicit formula for the powers of a generalized Beltrami

In this section, we will consider a generalized Beltrami

$$L=\rho\omega_{\bar{z}}+q\omega_z+\mu$$

i.e. $L=l+\mu$, $\mu=\mu(z,\bar{z})$ is differentiable function. Then,

$$L^n f=l^n f$$
$$+\sum_{p=1}^{n}\sum_{\substack{n_0,n_1,\dots,n_p\\ n_1+n_2+\cdots+n_p=n-p}} l^{n_0}(\mu l^{n_1}(\mu l^{n_2}\dots(\mu l^{n_p}f)))$$

Let $z_1=l^{n_1}\left(\mu l^{n_2}\left(\dots(\mu l^{n_p}f)\right)\right)$. Then,

$$l^{n_0}[\mu z_1]=\sum_{k_1=0}^{n_0}\binom{n_0}{k_1} l^{k_1}\mu l^{n_0-k_1}z_1, \quad l^{n_0-k_1}z_1$$
$$=l^{n_0-k_1}[l^{n_1}\mu z_2],$$

with $z_2=l^{n_2}\left(\mu l^{n_3}\left(\dots(\mu l^{n_p}f)\right)\right)$.

If $\rho = \text{const.}$,

$$l^{n_0-k_1}z_1 = l^{n_0+n_1-k_1}[\mu z_2]$$
$$= \sum_{k_2=0}^{n_0+n_1-k_1} \binom{n_0+n_1-k_1}{k_2} l^{k_2}\mu l^{n_0+n_1-k_1-k_2}z_2$$
$$= \sum_{k_2=0}^{n_0+n_1-k_1} \sum_{l_1=0}^{1} \binom{n_0+n_1-k_1}{k_2}\binom{1}{j_1} \times$$
$$\times\, l^{k_2}\mu^{j_1}.\, l^{n_0+n_1-k_1-k_2}\mu^{2-j_1}z_2, \qquad (3.1)$$

$$l^{n_0+n_1-k_1-k_2}u^{2-j_1}z_2 = l^{n_0+n_1+n_2-k_1-k_2}u^{2-j_1}[l^{n_2}\mu z_3]$$
$$= \sum_{k_3=0}^{n_0+n_1+n_2-k_1-k_2} \binom{n_0+n_1+n_2-k_1-k_2}{k_3} \times$$
$$\times\, l^{k_3}\mu l^{n_0+n_1+n_2-k_1-k_2-k_3}z_3.$$

By substituting in (3.1), we get

$$l^{n_0-k_1}z_1 = \sum_{k_2=0}^{n_0+n_1-k_1} \sum_{j_1=0}^{1} \binom{1}{j_1}\binom{n_0+n_1-k_1}{k_2} l^{k_2}\mu^{j_1}.\,\mu^{2-j_1}$$
$$\times \sum_{k_3=0}^{n_0+n_1+n_2-k_1-k_2} \sum_{j_2=0}^{2-j_1} \binom{2-j_1}{j_2}\binom{n_0+n_1+n_2-k_1-k_2}{k_3}$$
$$\times\, l^{k_3}\mu l^{n_0+n_1+n_2-k_1-k_2-k_3}\mu z_3$$

and hence,

$$L^n f = l^n f$$
$$+ \sum_{k_1=0}^{n_0} \sum_{k_2=0}^{n_0+n_1-k_1} \cdots \sum_{k_p=0}^{n_0+n_1+\cdots+n_p-k_1-\cdots-k_{p-1}} \sum_{j_1=0}^{1}\sum_{j_2=0}^{2-j_1}\cdots \sum_{j_p=0}^{p-j_1-j_2-\cdots-j_{p-1}} \binom{n_0}{k_1} \times$$
$$\times \binom{n_0+n_1+\cdots+n_p-k_1-\cdots-k_{p-1}}{k_p}\binom{1}{j_1}\binom{2-j_1}{j_2}\cdots\binom{p-j_1-j_2-\cdots-j_{p-1}}{j_p}$$
$$\times (l^{k_1}\mu)(l^{k_2}\mu^{j_1})\ldots(l^{k_p}\mu^{j_{p-1}})$$
$$(l^{n_0+n_1+\cdots+n_p-k_1-\cdots-k_{p-1}}\mu^{p-j_1-j_2-\cdots-j_{p-1}})f,$$

which can be written as

$$L^n f = l^n f$$
$$+ \sum_{\substack{k_1,k_2,k_3,\ldots,k_p\geq 0\\ k_1+k_2+\cdots+k_p=n-p}} \sum_{\substack{j_1,j_2,\cdots,j_p\geq 0\\ j_1+j_2+\cdots+j_p=p}} C^{n_0,n_1,\ldots,n_p}_{k_1,k_2,\ldots,k_p:j_1-j_2-\cdots-j_p}$$
$$\times (l^{k_1}\mu)(l^{k_2}\mu^{j_1})\ldots(l^{k_p}\mu^{j_{p-1}})$$
$$\times (l^{n_0+n_1+\cdots+n_p-k_1-\cdots-k_{p-1}}\mu^{p-j_1-j_2-\cdots-j_{p-1}})f,$$

where,

$$C^{n_0,n_1,\ldots,n_p}_{k_1,k_2,\ldots,k_p:j_1,j_2,\ldots,j_p} = \binom{n_0}{k_1}\binom{n_0+n_1+\cdots+n_p-k_1-\cdots-k_{p-1}}{k_p} \times$$
$$\times \binom{1}{j_1}\binom{2-j_1}{j_2}\ldots\binom{p-j_1-j_2-\cdots-j_{p-1}}{j_p}.$$

3.2. Alternative formula for powers of a generalized Beltrami operator

Theorem 4. The n^{th}- power $(n \geq 2)$ of the generalized Beltrami operator L is given as

$$L^n = l^n + \sum_{k=1}^{n} \binom{n}{k} B_{k,l}[\mu_0, \mu_1, \ldots, \mu_n], \qquad (3.2)$$

where $\mu_k = l^k\mu$, with $l = \rho\partial_{\bar{z}} + q\partial_z$ is the Beltrami operator.

Proof. The obvious product rule

$$L[\omega_1\omega_2] = [l+\mu][\omega_1\omega_2]$$
$$= l[\omega_1\omega_2] + \mu\omega_1\omega_2$$
$$= l[\omega_1]\omega_2 + \omega_1 l[\omega_2] + \mu\omega_1\omega_2$$
$$= [l+\mu][\omega_1]\omega_2 + \omega_1 l[\omega_2]$$
$$= L[\omega_1]\omega_2 + \omega_1 l[\omega_2],$$
$$L^2[\omega_1\omega_2] = L(L[\omega_1\omega_2])$$
$$= L[L[\omega_1]\omega_2 + \omega_1 l[\omega_2]]$$
$$= L(L[\omega_1]\omega_2) + L(\omega_1 l[\omega_2])$$
$$= L^2[\omega_1]\omega_2 + 2L[\omega_1]l[\omega_1] + \omega_1 l^2[\omega_2],$$
$$= \sum_{k=0}^{2} \binom{2}{k} L^k[\omega_1]\, l^{2-k}[\omega_2].$$

Generally, we have

$$L^n[\omega_1\omega_2] = \sum_{k=0}^{n} \binom{n}{k} [L^k\omega_1]\, l^{n-k}[\omega_2]. \qquad (3.3)$$

Specializing here $\omega_1 = 1,\ \omega_2 = \mu$, we obtain the recursion

$$L^n\mu = \textstyle\sum_{k=0}^{n} \binom{n}{k} [L^k 1]\, l^{n-k}[\mu]. \qquad (3.4)$$

Thus, $L^n\mu$ is known for every n if $L^k 1$ is known for every k. Since $\mu = L[1]$, then

$$L^{n+1}[1] = \sum_{k=0}^{n} \binom{n}{k} L^k[1]\, l^{n-k}[\mu], \qquad (3.5)$$
$$L^0 1 = 1. \qquad (3.5')$$

Comparing $(3.5),(3.5')$ with $(2.2),(2.2')$, gives

$$L^k 1 = B_k^l[\mu, l\mu, \dots, l^n\mu] = B_k[\mu_0, \mu_1, \dots, \mu_n],$$

$$\mu_k = l^k \mu.$$

4. Induction proof of Theorem 4

The basic results in Theorems 3 and 4 can be proven by mathematical induction. To avoid the monotony of repetition, we will present the induction proof of Theorem 4.

For $n = 1$, $L^1 = L = l + u$. This can be expressed as:

$$l^1 + \sum_{k=1}^{1} \binom{1}{k} B_{\{1,k\}} = l + B_{\{1,1\}} = l + 1.u,$$

where $B_{\{1,1\}} = u$. Thus $L^1 = l + u = l^1 + B_{\{1,1\}}$. This means that the base case holds.

Now, for $n = k$ assume that

$$L^k = l^n + \sum_{j=1}^{k} \binom{k}{j} B_{\{k,j\}}$$

For $n = k + 1$, $L^{k+1} = L(L^k)$. Using the inductive hypothesis, one gets

$$L^{k+1} = L\left(l^k + \sum_{j=1}^{k} \binom{k}{j} B_{\{k,j\}} \right),$$

This can be rewritten as

$$L^{k+1} = L(l^k) + L\left(\sum_{j=1}^{k} \binom{k}{j} B_{\{k,j\}} \right). \qquad (4.1)$$

Using the linearity of L, applying L to l^k, gives

$$L(l^k) = l(l^k) + u(l^k).$$

The term $u(l^k)$ is simply, l^k. While the term $l(l^k)$ can be computed as follows:

$$l(l^k) = \partial_{\bar{z}}(l^k) + q\,\partial_z(l^k).$$

Now consider the contribution from the second term in (4.1)

$$L\left(\sum_{j=1}^{k} \binom{k}{j} B_{\{k,j\}} \right) = \sum_{j=1}^{k} \binom{k}{j} L(B_{\{k,j\}})$$

with $LB_{\{k,j\}} = lB_{\{k,j\}} + uB_{\{k,j\}}$.

Thus, collecting terms of (4.1), we have

$$L^{k+1} = l(l^k) + u(l^k) + \sum_{j=1}^{k} \binom{k}{j} (lB_{\{k,j\}} + uB_{\{k,j\}}).$$

We can group the terms involving l to get

$$L^{k+1} = l^{k+1} + \sum_{j=1}^{k} \binom{k}{j} l(B_{\{k,j\}})$$

$$+ \sum_{j=1}^{k} \binom{k}{j} uB_{\{k,j\}} + ul^k. \qquad (4.2)$$

By the properties of Bell polynomials, we recognize the contributions of $l(B_{\{k,j\}})$.

We can express

$$l(B_{\{k,j\}}) = \sum_{m=0}^{j} \binom{j}{m} B_{\{k+1,m\}}$$

leading to the sum:

$$\sum_{j=1}^{k} \binom{k}{j} l(B_{\{k,j\}}) = \sum_{m=0}^{k} (\sum_{j=m}^{k} \binom{k}{j}\binom{j}{m}) B_{\{k+1,m\}}.$$

The combinatorial identity gives

$$\sum_{j=m}^{k} \binom{k}{j}\binom{j}{m} = \binom{k+1}{m+1}.$$

Finally, putting everything together, Eq.(4.2) is of the form

$$L^{k+1} = l^{k+1} + \sum_{j=m}^{k} \binom{k+1}{m} B_{\{k+1,m\}}.$$

That is, the induction hypothesis holds for $n = k + 1$. Thus, we conclude that:

$$L^n = l^n + \sum_{k=1}^{n} \binom{n}{k} B_{\{n,k\}}, \quad n \geq 1.$$

This completes the proof of Theorem 4.

5. Applications

Here, we introduce an application for the Bell polynomial for the Beltrami operator. It generalizes the Faa' di Bruno formula. It is diverse for the obtained formula to be called a Beltrami- Faa' di Bruno formula. Moreover, some illustrative examples are given.

Let $g = g(z,\bar{z})$ and $f(z)$ be complex valued functions. Apply l^n to the composition function $f \circ g$, leads to the Beltrami- Faa' di Bruno formula:

$$l^n\big(f \circ g(z,\bar{z})\big) = \sum_{j=0}^{n} f^{(j)}(g(z,\bar{z}))B_{n,j}(g_z, g_{\bar{z}}, g_{zz}, g_{z\bar{z}}, g_{\bar{z}\bar{z}}, \dots),$$

where

$$g_z = \frac{\partial g}{\partial z}, g_{\bar{z}} = \frac{\partial g}{\partial \bar{z}}, g_{zz} = \frac{\partial^2 g}{\partial z^2}, g_{z\bar{z}} = \frac{\partial^2 g}{\partial z \partial \bar{z}},$$

and $g_{\bar{z}\bar{z}} = \frac{\partial^2 g}{\partial \bar{z}^2}$.

Each term in the sum accounts for the contributions from the derivatives of f evaluated at g and the corresponding Bell polynomial of the derivatives of g.

5.1. Example 1

If $g = e^{(z\bar{z})}$ and $f = \frac{1}{(z+1)}$, one can obtain

$$l^n(f \circ g) = \sum_{j=0}^{n} \frac{(-1)^j j!}{(e^{z\bar{z}}+1)^{j+1}} B_{n,j}(\bar{z}e^{z\bar{z}}, ze^{z\bar{z}}, \bar{z}^2 e^{z\bar{z}}, e^{z\bar{z}} + z\bar{z}e^{z\bar{z}}, z^2 e^{z\bar{z}}, \dots)$$

with

$$B_{n,j}(\bar{z}e^{z\bar{z}}, ze^{z\bar{z}}, \bar{z}^2 e^{z\bar{z}}, e^{z\bar{z}} + z\bar{z}e^{z\bar{z}}, z^2 e^{z\bar{z}}, \dots) = e^{nz\bar{z}} \sum_{k_1+k_2+\cdots+k_n=n} \frac{n!}{k_1!\,k_2!\cdots k_n!} \prod_{i=1}^{j} x_i^{k_i}$$

where

$$x_1 = \bar{z},\ x_2 = z,\ x_3 = \bar{z}^2,\ x_4 = 1 + z\bar{z},\ x_5 = z^2, \cdots$$

5.2. Example 2 An expression for $l^n(f \circ g)$ where $f(z) = \frac{1}{z}$ and $g(z) = \ln(z+1)$ using Bell polynomials can be calculated as

$$l^n\big(f \circ g(z)\big) = \sum_{j=0}^{n} \left((-1)^j \frac{(j-1)!}{(\ln(z+1))^j}\right) B_{n,j}\left(\frac{1}{z+1}, -\frac{1}{(z+1)^2}, \dots\right)$$

with

$$B_{n,j}\left(\frac{1}{z+1}, -\frac{1}{(z+1)^2}, -\frac{1}{(z+1)^3}, \dots\right) = \sum_{k_1+k_2+\cdots+k_j=n} \frac{n!\,(-1)^{k_2+k_3+\cdots+k_j}}{k_1!\,k_2!\cdots k_j!\,(z+1)^{k_1+2k_2+3k_3+\cdots+jk_j}}$$

Conclusion

In this study, we have introduced generalized Bell polynomials and investigated basic properties of these polynomials. We have obtained Bell polynomials for the Beltrami operator. Also, we got explicit formulas for the powers of the Beltrami operator and a generalized Beltrami operator. We introduced an application of the obtained Bell polynomial for the Beltrami operator, namely, a Beltrami- Faà di Bruno formula had been established. Illustrative examples were given. Further, the obtained explicit formulas for the powers of the Beltrami operator and generalized Beltrami operators can help in tackling new application in the conformal mapping theory and boundary value problem for elliptic complex differential operators of higher order involving Beltrami and generalized Beltrami operators of higher orders. That we are going to do in future work.

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