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Research Article:

Investigation of Soliton Solutions for Some Important Nonlinear Evolution Equations Via Exp(-Φ(ζ)) Expansion Method

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Abstract

One effective technique for finding some exact traveling wave solutions of nonlinear partial differential equations (NPDEs) is the $\exp(-\Phi(\zeta))$ expansion method. In this paper, the exact traveling wave solutions for the nonlinear coupled Whitham-Broer-Kaup equation and the new coupled Korteweg-de Vries (KdV) equation are obtained by applying the $\exp(-\Phi(\zeta))$ expansion method. The numerical results of these solutions by using Maple have been presented graphically and discussed. The obtained traveling wave solutions include exponential functions, hyperbolic functions, trigonometric functions, and rational functions. Moreover, 3D graphics of solutions like the bell-shaped soliton solution, kink-type, periodic traveling waves, singular kink-type, singular cuspon type, as well as plane-wave solutions are presented to illustrate the dynamics of the equations. Comparing the results of the proposed method with the results of the homotopy analysis method shows that the proposed method is a strong and attractive method for solving systems of nonlinear partial differential equations. The results demonstrated the efficiency and simplicity of this method in extracting these exact solutions. The effectiveness of this method in solving nonlinear coupled partial differential equations that arise in mathematical physics and engineering has been shown.

1. Introduction

Recently, nonlinear evolution equations (NLEEs) have played a significant role in interpreting many nonlinear phenomena. Nonlinear phenomena emerge in a variety

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of scientific applications, such as plasma physics, solid-state heat flow, wave propagation phenomena, quantum mechanics, fluid mechanics, shallow water wave propagation, and so on.

Through exact solutions, we can better understand and explain the phenomena modeled by NLEEs. Great efforts have been made to find exact solutions to such nonlinear equations by many mathematicians and physicists. Many effective techniques have been developed to generate more new ideas. One of the most important of these techniques is the use of wave variable transformation to transform NPDE to ordinary differential equation (ODE) to obtain the solitons. In reality, no single method can be utilized for all kinds of nonlinear evolution equations. There are various techniques for soliton solutions, including inverse scattering method (Ablowitz & Clarkson, 1991), Bäcklund and Darboux transformation method (Rogers & Schief, 2002), homotopy perturbation method (He, 1998; Mohyud & Noor, 2009; Rashidi et al., 2008; Abbasbandy, 2007), first integral method (Taghizadeh et al., 2012), variational iteration method (Yusufoğlu, 2008), Riccati-Bernoulli sub-ODE method (Yang et al., 2015), Jacobi elliptic function method (Yan, 2003), tanh-sech method (Wazwaz, 2005), (G'/G)-expansion method (Wang et al., 2008; Zhang, 2009), Hirota's method (Hirota, 1971), homogeneous balance method (M.L, 1995; Wang & Zhou, 1996; Rady et al., 2010; Rady et al., 2009; Khalfallah, 2009), $\exp(-\phi(\zeta))$ expansion method (Akbar & Norhashidah, 2014; Hafez et al., 2014; Elboree, 2021), differential transform method (DTM) (Chen & Ho, 1999; Fatoorehchia & Abolghasemi, 2014; Fatoorehchi & Abolghasemi, 2013; Kurnaz et al., 2005)and so on.

A solitary wave with elastic scattering is called a soliton. Solitons maintain their forms and velocity after interacting with one another. Solitary waves are described by the KdV equation, while shock waves are governed by the KdV-Burgers equation. A delicate balance between dispersive and nonlinear effects in the medium leads to solitons. Solitons take the form of a kink or the sech^2 bell shape.

In previous studies, the $\exp(-\Phi(\zeta))$ expansion method was not applied to the aforementioned equations, but other methods were considered, including the homotopy analysis method. Through the homotopy analysis method, S. Abbasbandy (Abbasbandy, 2007) obtained traveling wave solutions for the coupled KdV equation. Also, M. M. Rashidi, D. D. Ganji, and S. Dinarvand (Rashidi et al., 2008) derived traveling wave solutions for the coupled Whitham-Broer-Kaup equations. The results of the homotopy analysis method and the suggested method were compared. According to the results, one of the advantages of the $\exp(-\Phi(\zeta))$ expansion method is that it offers new exact solutions for the traveling wave along with additional free parameters. Soliton, periodic traveling wave solution, kink, and cuspons are obtained when the relevant physical parameters are given their respective values. With Maple, the algebraic manipulation of the suggested scheme is much simpler than previous methods.

Some new analytical solutions have been provided by $\exp(-\Phi(\zeta))$ expansion method instead of other ways. Thus, the objective of this paper is to acquire the exact traveling wave solutions of the coupled Whitham-Broer-Kaup equation and the new coupled KdV equation using $\exp(-\Phi(\zeta))$ expansion method.

The arrangement of the article is as follows: Section 2 describes the $\exp(-\Phi(\zeta))$ expansion method. The traveling wave solutions of the coupled Whitham-Broer-Kaup equation and the new coupled KdV equation are obtained in Section 3. Section 4 displays the physical explanations and graphical representations of the solutions. Section 5 presents the comparison. Finally, conclusions are drawn in the last section.

2. The $\exp(-\Phi(\zeta))$ expansion method

This section describes the $\exp(-\Phi(\zeta))$ expansion method to obtain traveling wave solutions of NLEEs. The NPDE's general form can be expressed as follows:

$$R\left(v, v_t, v_x, v_y, v_z, v_{xx}, v_{yy}, v_z, v_{tt}, v_{tx}, v_{ty}, v_{tz}, \ldots\right) = 0 \quad (1)$$

where $v(x, y, z, t)$ is an unknown function, R is a polynomial in v (x, y, z, t) and its derivatives. Here are the basic steps of the proposed technique:

Step 1: We combine the real variables x, y, z and $t$by a complex variable ζ is as follows:

$$v(x,\ y,\ z,\ t) = V(\zeta),\ \zeta = x + y + z - ct \quad (2)$$

where c is the wave speed. Using the traveling wave transformation (2), we obtain the ODE of Eq. (1) for V = V(ζ)

$$R(V, V', V'', V''', \ldots\ldots) = 0 \quad (3)$$

where R is a polynomial of $V(\zeta)$ and its derivatives and the superscripts indicate the ordinary derivatives with respect to ζ.

Step 2: According to the $\exp(-\Phi(\zeta))$ expansion method, the following is a formula for the traveling wave solution of Eq. (3):

$$V(\zeta) = \sum_{j=0}^{N} a_j \exp(-\Phi(\zeta))^j \quad (4)$$

where a_j are constants to be determined later and $\Phi(\zeta)$ satisfies the subsequent ODE:

$$\Phi'(\zeta) = \exp(-\Phi(\zeta)) + \tau\exp(\Phi(\zeta)) + \kappa \quad (5)$$

where τ and κ are arbitrary constants. It is worth noting that Eq. (5) has the subsequent generic solutions:

Type 1: If $\tau \neq 0$ and $\kappa^2 - 4\tau > 0$

$$\Phi(\zeta) = \ln\left(-\frac{\sqrt{\kappa^2-4\tau}\tanh\left(\sqrt{\frac{(\kappa^2-4\tau)}{2}}(D+\zeta)\right)+\kappa}{2\tau}\right) \quad (6)$$

Type 2: If $\tau \neq 0$ and $\kappa^2 - 4\tau < 0$,

$$\Phi(\zeta) = \ln\left(\frac{\sqrt{4\tau-\kappa^2}\tan\left(\sqrt{\frac{(4\tau-\kappa^2)}{2}}(D+\zeta)\right)-\kappa}{2\tau}\right) \quad (7)$$

Type 3: If $\tau = 0$, $\kappa \neq 0$ and $\kappa^2 - 4\tau > 0$,

$$\Phi(\zeta) = -\ln\left(\frac{\kappa}{\cosh(\kappa(D+\zeta))+\sinh(\kappa(D+\zeta))-1}\right) = -\ln\left(\frac{\kappa}{\exp(\kappa(D+\zeta))-1}\right) \quad (8)$$

Type 4: If $\tau \neq 0$, $\kappa \neq 0$ and $\kappa^2 - 4\tau = 0$,

$$\Phi(\zeta) = \ln\left(-\frac{2(\kappa(\zeta+D)+2)}{\kappa^2(D+\zeta)}\right) \quad (9)$$

Type 5: If $\tau = 0$, $\kappa = 0$ and $\kappa^2 - 4\tau = 0$,

$$\Phi(\zeta) = \ln(\zeta + D) \quad (10)$$

where D is an integration constant.

Step 3: Obtaining the positive integer N requires achieving a balance between the nonlinear terms of the highest order and the highest-order derivatives that appeared in Eq. (3).

Step 4: By substituting Eq. (4) into Eq. (3) and utilizing (5), we will get a system of algebraic equations by setting all the coefficients of $\exp(-\Phi(\zeta))$ to zero. We solve the obtained system to get the values of a_n, c, κ, and τ through symbolic computation software such as Maple.

Step 5: Replacing values of these constants into Eq. (4) along with the general solutions of Eq. (5), we obtain the solution of Eq. (1).

3. Application of the method

In this part, we will apply the $\exp(-\Phi(\zeta))$ expansion method to get new traveling wave solutions for the nonlinear coupled Whitham-Broer-Kaup equation and the new coupled KdV equation, both of which represent significant NLEEs in mathematical physics and engineering.

A. The coupled Whitham-Broer-Kaup equation: Applying the $\exp(-\Phi(\zeta))$ expansion method, we can solve the coupled Whitham-Broer-Kaup equation presented by Whitham, Broer, and Kaup exactly. The equations describe the propagation of shallow water waves with different propagation relations. Let us consider the coupled Whitham-Broer-Kaup equation:

$$u_t + uu_x + v_x + \beta u_{xx} = 0,$$

$$v_t + (uv)_x - \beta v_{xx} + \gamma u_{xxx} = 0 \quad (11)$$

Where $u = u\ (x,t)$ represents the horizontal velocity, $v= v\ (x,t)$ is the height that deviates from the equilibrium position of the liquid, and β, γ are constants that represent different diffusion power. The traveling wave variable is employed

$$u(x,t) = U(\zeta), v(x,t) = V(\zeta), \zeta = x - ct \quad (12)$$

Eq. (12) is reduces Eq. (11) into ODEs as follows:

$$-cU' + UU' + V' + \beta U'' = 0, \quad (13)$$

$$-cV' + [UV]' - \beta V'' + \gamma U^{(3)} = 0 \quad (14)$$

When Eqs. (13) and (14) are integrated with regard to ζ and considering the integration constant to be zero, yields

$$-cU + \tfrac{1}{2}U^2 + V + \beta U' = 0 \quad (15)$$

$$-cV + UV - \beta V' + \gamma U'' = 0 \quad (16)$$

By applying the homogeneous balance between the highest order derivatives and nonlinear terms in Eqs. (15) and (16), we conclude that N = 1 in Eq. (15) and N = 2 in Eq. (16). The $\exp(-\Phi(\zeta))$ expansion method enables us to use the solution as follows:

$$U(\zeta) = a_0 + a_1 \exp(-\Phi(\zeta)) \quad (17)$$

$$V(\zeta) = b_0 + b_1 \exp(-\Phi(\zeta)) + (\exp(-\Phi(\zeta)))^2 \quad (18)$$

where a_j and b_j are constants to be determined. By substituting (17) and (18) into (15) and (16), then equating the coefficients of $\exp(-\Phi(\zeta))$ to zero, we obtain a set of algebraic equations for the parameters a_0, a_1, b_0, b_1, τ, κ and β. Solving this system we get

$$\beta = \beta, c = c, a_0 = a_0, a_1 = a_1,$$

$$b_0 = \beta a_1 \tau + c a_0 - \frac{1}{2}a_0^2,$$

$$b_1 = \beta\kappa a_1 + c a_1 - a_0 a_1,$$

$$b_2 = \beta a_1 - \tfrac{1}{2}a_1^2 \quad (19)$$

where τ and κ are arbitrary constants.

Utilizing Eqs. (17), (18), and (19), we obtained the traveling wave solutions for the coupled Whitham-Broer-Kaup equation according to the formulas Eqs. (6)-(10) as follows

When $\tau \neq 0$ and $\kappa^2 - 4\tau > 0$, we have

$$u_1(x,t) = a_0 - \frac{2a_1\tau}{\sqrt{\kappa^2 - 4\tau}\tanh\left(\frac{1}{2}\sqrt{2\kappa^2 - 8\tau}(D + x - ct)\right) + \kappa} \quad (20)$$

$$v_1(x,t) = b_0 - \frac{2b_1\tau}{\sqrt{\kappa^2 - 4\tau}\tanh\left(\frac{1}{2}\sqrt{2\kappa^2 - 8\tau}(D + x - ct)\right) + \kappa} + \frac{4b_2\tau^2}{(\sqrt{\kappa^2 - 4\tau}\tanh(\frac{1}{2}\sqrt{2\kappa^2 - 8\tau}(D + x - ct)) + \kappa)^2} \quad (21)$$

When $\tau \neq 0$ and $\kappa^2 - 4\tau < 0$, we have

$$u_2(x,t) = a_0 + \frac{2a_1\tau}{\sqrt{4\tau - \kappa^2}\tan(\frac{1}{2}\sqrt{8\tau - 2\kappa^2}(D + x - ct)) - \kappa} \quad (22)$$

$$v_2(x,t) = b_0$$

$$+\frac{2b_1\tau}{(\sqrt{4\tau-\kappa^2}\tan(\frac{1}{2}\sqrt{8\tau-2\kappa^2}(D+x-ct))-\kappa}$$
$$+\frac{4b_2\tau^2}{(\sqrt{4\tau-\kappa^2}\tan(\frac{1}{2}\sqrt{8\tau-2\kappa^2}(D+x-ct))-\kappa)^2} \quad (23)$$

When $\tau = 0$, $\kappa \neq 0$ and $\kappa^2 - 4\tau > 0$, we have

$$u_3(x,t) = a_0 + \frac{a_1\kappa}{\exp(\kappa(D+x-ct))-1} \quad (24)$$

$$v_3(x,t) = b_0 + \frac{b_1\kappa}{\exp(\kappa(D+x-ct))-1} + \frac{b_2\kappa^2}{(\exp(\kappa(D+x-ct))-1)^2} \quad (25)$$

When $\tau \neq 0$, $\kappa \neq 0$ and $\kappa^2 - 4\tau = 0$, we have

$$u_4(x,t) = a_0 - \frac{a_1\kappa^2(D+x-ct)}{2\kappa(D+x-ct)+4} \quad (26)$$

$$v_4(x,t) = b_0 - \frac{b_1\kappa^2(D+x-ct)}{2\kappa(D+x-ct)+4} + \frac{b_2\kappa^4(D+x-ct)^2}{(2\kappa(D+x-ct))+4)^2} \quad (27)$$

When $\tau = 0$, $\kappa = 0$ and $\kappa^2 - 4\tau = 0$, we have

$$u_5(x,t) = a_0 + \frac{a_1}{D+x-ct} \quad (28)$$

$$v_5(x,t) = b_0 + \frac{b_1}{D+x-ct} + \frac{b_2}{(D+x-ct)^2} \quad (29)$$

where D is an integration constant.

B. The new coupled KdV equation: The following part will introduce the $\exp(-\Phi(\zeta))$ expansion method to obtain the exact solutions of the new coupled KdV model equations that explain the interactions between two long waves with various dispersion relations. These equations are in the following form:

$$u_t + \frac{1}{4}u_{xxx} - \frac{3}{2}uu_x - \frac{3}{4}(vw)_x = 0,$$
$$v_t + \frac{1}{4}v_{xxx} - \frac{3}{2}(uv)_x = 0,$$
$$w_t + \frac{1}{4}w_{xxx} - \frac{3}{2}(uw)_x = 0 \quad (30)$$

Now let us assume that the traveling wave transformation equation is

$$u(x,t) = U(\zeta), v(x,t) = V(\zeta),$$
$$w(x,t) = W(\zeta), \zeta = x - ct \quad (31)$$

Using traveling wave transformation, Eq. (30) is carried to an ODE

$$-cU' + \frac{1}{4}U^{(3)} - \frac{3}{2}UU' - \frac{3}{4}[VW]' = 0, \quad (32)$$

$$-cV' + \frac{1}{4}V^{(3)} - \frac{3}{2}[UV]' = 0, \quad (33)$$

$$-cW' + \frac{1}{4}W^{(3)} - \frac{3}{2}[UW]' = 0. \quad (34)$$

By integrating Eqs. (32), (33) and (34) with respect to ζ, and choosing the constant of integration as zero, we obtain

$$-cU + \frac{1}{4}U^{(2)} - \frac{3}{4}U^2 - \frac{3}{4}VW = 0, \quad (35)$$

$$-cV + \frac{1}{4}V^{(2)} - \frac{3}{2}UV = 0, \quad (36)$$

$$-cW + \frac{1}{4}W^{(2)} - \frac{3}{2}UW = 0 \quad (37)$$

Taking the homogeneous balance between the highest order derivatives and nonlinear terms in Eqs. (35), (36) and (37), we obtain that N = 2 in Eq. (35), N = 2 in Eq. (36), and N = 2 in Eq. (37). Thus, the solution of Eqs. (35)-(37) takes the following form:

$$U(\zeta) = a_0 + a_1\exp(-\Phi(\zeta)) + a_2(\exp(-\Phi(\zeta)))^2 \quad (38)$$

$$V(\zeta) = b_0 + b_1\exp(-\Phi(\zeta)) + b_2(\exp(-\Phi(\zeta)))^2 \quad (39)$$

$$W(\zeta) = d_0 + d_1\exp(-\Phi(\zeta)) + d_2(\exp(-\Phi(\zeta)))^2 \quad (40)$$

where a_j, b_j, and d_j are constants to be determined. Substituting Eqs. (38)-(40) into Eq. (35), (36), and (37), then setting the coefficients of $\exp(-\Phi(\zeta))$ to zero, we get a set of algebraic equations for the parameters a_0,

$a_1, a_2, b_0, b_1, b_2, d_0, d_1, d_2, \tau$ and κ . By solving this system, we yield

$$a_0 = \frac{1}{3}\tau - \frac{2}{3}c, a_1 = \frac{1}{3}, a_2 = \frac{1}{3}, b_0 = b_0,$$
$$b_1 = \frac{2b_0}{\kappa}, b_2 = 0, d_0 = d_0, d_1 = d_1, d_2 = d_2 \quad (41)$$

By substituting Eqs. (6)-(10) into Eqs. (38)-(40), we get the following traveling wave solutions of the new coupled KdV model equations.

When $\tau \neq 0$ and $\kappa^2 - 4\tau > 0$, we have

$$u_6(x,t) = a_0 - \frac{2a_1\tau}{\sqrt{\kappa^2-4\tau}\tanh\left(\frac{1}{2}\sqrt{2\kappa^2-8\tau}(D+x-ct)\right)+\kappa} + \frac{2a_2\tau^2}{\left(\sqrt{\kappa^2-4\tau}\tanh\left(\frac{1}{2}\sqrt{2\kappa^2-8\tau}(D+x-ct)\right)+\kappa\right)^2} \quad (42)$$

$$v_6(x,t) = b_0 - \frac{2b_1\tau}{\sqrt{\kappa^2-4\tau}\tanh\left(\frac{1}{2}\sqrt{2\kappa^2-8\tau}(D+x-ct)\right)+\kappa} + \frac{4b_2\tau^2}{\left(\sqrt{\kappa^2-4\tau}\tanh\left(\frac{1}{2}\sqrt{2\kappa^2-8\tau}(D+x-ct)\right)+\kappa\right)^2} \quad (43)$$

$$w_6(x,t) = d_0 - \frac{2d_1\tau}{\sqrt{\kappa^2-4\tau}\tanh\left(\frac{1}{2}\sqrt{2\kappa^2-8\tau}(D+x-ct)\right)+\kappa} + \frac{4d_2\tau^2}{\left(\sqrt{\kappa^2-4\tau}\tanh\left(\frac{1}{2}\sqrt{2\kappa^2-8\tau}(D+x-ct)\right)+\kappa\right)^2} \quad (44)$$

When $\tau \neq 0$ and $\kappa^2 - 4\tau < 0$, we have

$$u_7(x,t) = a_0 + \frac{2a_1\tau}{\sqrt{4\tau-\kappa^2}\tan\left(\frac{1}{2}\sqrt{8\tau-2\kappa^2}(D+x-ct)\right)-\kappa} + \frac{4a_2\tau^2}{\left(\sqrt{4\tau-\kappa^2}\tan\left(\frac{1}{2}\sqrt{8\tau-2\kappa^2}(D+x-ct)\right)-\kappa\right)^2} \quad (45)$$

$$v_7(x,t) = b_0 + \frac{2b_1\tau}{\sqrt{4\tau-\kappa^2}\tan\left(\frac{1}{2}\sqrt{8\tau-2\kappa^2}(D+x-ct)\right)-\kappa} + \frac{4b_2\tau^2}{\left(\sqrt{4\tau-\kappa^2}\tan\left(\frac{1}{2}\sqrt{8\tau-2\kappa^2}(D+x-ct)\right)-\kappa\right)^2} \quad (46)$$

$$w_7(x,t) = d_0 + \frac{2d_1\tau}{\sqrt{4\tau-\kappa^2}\tan\left(\frac{1}{2}\sqrt{8\tau-2\kappa^2}(D+x-ct)\right)-\kappa} + \frac{4d_2\tau^2}{\left(\sqrt{4\tau-\kappa^2}\tan\left(\frac{1}{2}\sqrt{8\tau-2\kappa^2}(D+x-ct)\right)-\kappa\right)^2} \quad (47)$$

When $\tau = 0$, $\kappa \neq 0$ and $\kappa^2 - 4\tau > 0$, we have

$$u_8(x,t) = a_0 + \frac{a_1\kappa}{\exp(\kappa(D+x-ct)-1)} + \frac{a_2\kappa^2}{(\exp(\kappa(D+x-ct))-1)^2} \quad (48)$$

$$v_8(x,t) = b_0 + \frac{b_1\kappa}{\exp(\kappa(D+x-ct)-1)} + \frac{b_2\kappa^2}{(\exp(\kappa(D+x-ct))-1)^2} \quad (49)$$

$$w_8(x,t) = d_0 + \frac{d_1\kappa}{\exp(\kappa(D+x-ct)-1)} + \frac{d_2\kappa^2}{(\exp(\kappa(D+x-ct))-1)^2} \quad (50)$$

When $\tau \neq 0$, $\kappa \neq 0$ and $\kappa^2 - 4\tau = 0$, we have

$$u_9(x,t) = a_0 - \frac{a_1\kappa^2(D+x-ct)}{2\kappa(D+x-ct)+4} + \frac{a_2\kappa^4(D+x-ct)^2}{(2\kappa(D+x-ct)+4)^2} \quad (51)$$

$$v_9(x,t) = b_0 - \frac{b_1\kappa^2(D+x-ct)}{2\kappa(D+x-ct)+4} + \frac{b_2\kappa^4(D+x-ct)^2}{(2\kappa(D+x-ct)+4)^2} \quad (52)$$

$$w_9(x,t) = d_0 - \frac{d_1\kappa^2(D+x-ct)}{2\kappa(D+x-ct)+4} + \frac{d_2\kappa^4(D+x-ct)^2}{(2\kappa(D+x-ct)+4)^2}. \quad (53)$$

When $\tau = 0$, $\kappa = 0$ and $\kappa^2 - 4\tau = 0$, we have

$$u_{10}(x,t) = a_0 + \frac{a_1}{D+x-ct} + \frac{a_2}{(D+x-ct)^2} \quad (54)$$

$$v_{10}(x,t) = b_0 + \frac{b_1}{D+x-ct} + \frac{b_2}{(D+x-ct)^2} \quad (55)$$

$$w_{10}(x,t) = d_0 + \frac{d_1}{D+x-ct} + \frac{d_2}{(D+x-ct)^2} \quad (56)$$

where D is an integration constant.

4. Physical significance

In this part, we study the physical interpretations and graphical representation of traveling wave solutions of the coupled Whitham-Broer-Kaup equation and the new coupled KdV equation. It is worth noting that the delicate balance between the nonlinearity effect and the dissipative effect gives rise to solitons that maintain their speed and shape when interacting with others completely. This indicates that following nonlinear interaction, the amplitude, velocity, and wave shape of solitons remain unchanged and display fully elastic collisions. Therefore, the elastic property describes the physically traveling wave solutions, and these solutions are shown graphically in Figures 1-9.

A. The Coupled Whitham-Broer-Kaup equation

This part presents the physical interpretation of the traveling wave solutions of the coupled Whitham-Broer-Kaup equation. It is observed that the coupled Whitham-Broer-Kaup equation gives several traveling wave solutions in the x-y plane, including kink-type, periodic traveling waves, singular cuspon type as well as plane-waves solutions. Fig. 1 shows the wave profiles corresponding to the solutions u_1 and v_1 of the coupled Whitham-Broer-Kaup equation for $a_0 = 1$, $a_1 = 1$, $\tau = 1$, $\kappa = 3$, c = 5, D = 1, $b_0 = \frac{11}{2}$, $b_1 = 7$, $b_2 = \frac{1}{2}$, y = 0, z = 0 and $-10 \le x \le 10$, $-10 \le t \le 10$. It is noticed that in the x-y plane, the traveling wave solutions u_1 and v_1 initially represent kink-type soliton. Traveling waves that emerge from one asymptotic state to another are known as kink waves. This soliton is classified as a topological soliton. Furthermore, after the interactions, it is noted that the waves' amplitude and shape gradually decrease. They drop to zero in the final condition, indicating that the solitonic excitations are not fully elastic.

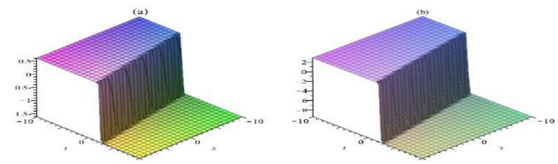

Figure 1: 3-D plot of the kink traveling wave solutions of u_1 and v_1 with $-10 \le x \le 10$, $-10 \le t \le 10$ respectively.

Fig. 2 presents the wave profiles corresponding to the solutions u_2 and v_2 for $a_0 = 1$, $a_1 = 1$, $\tau = \frac{1}{2}$, $\kappa = 1$, c = 1, D = 2, $b_0 = 1$, $b_1 = 1$, $b_2 = \frac{1}{2}$, y = 0, z = 0 and $-10 \le x \le 10$, $-10 \le t \le 10$. The solutions u_2 and v_2 form the exact periodic traveling wave solutions of the Whitham-Broer-Kaup equation in the x-y plane. It turns out that the solutions of periodic waves do not change. That is, after the nonlinear interaction, the solution of the periodic wave is fully flexible.

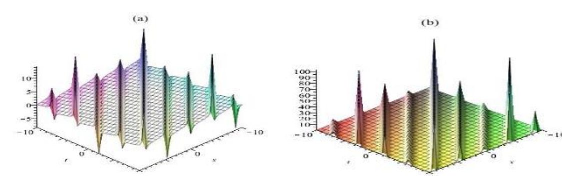

Figure 2: 3-D plot of the periodic traveling wave solutions of u_2 and v_2 with $-10 \le x \le 10$, $-10 \le t \le 10$ respectively.

The exact solutions u_3, u_4, u_5, and v_4 represent the plane-waves solutions of the coupled Whitham-Broer-Kaup equation in the x-y plane. Shape of Eq. (24) (u_3) with $a_0 = 1$, $a_1 = 1$, $\tau = 0$, $\kappa = 1$, c = 2, D = 2, y = 0, z = 0. Eq. (26) (u_4) with $a_0 = 1$, $a_1 = 1$, $\tau = 1$, $\kappa = 2$, c = 2, D = 5, y = 0, z = 0. Eq. (27) (v_4) with $b_0 = \frac{11}{2}$, $b_1 = 6$, $b_2 = \frac{1}{2}$, $\tau = 1$, $\kappa = 2$, c = 5, $D = \frac{1}{2}$, y = 0, z = 0, and Eq. (28) (u_5) with

$a_0 = 1$, $a_1 = 1$, $\tau = 0$, $\kappa = 0$, c = 3, D = 10, y = 0, z = 0 into the interval $-10 \leq x \leq 10$, $-10 \leq t \leq 10$ (see Figs. 3-5).

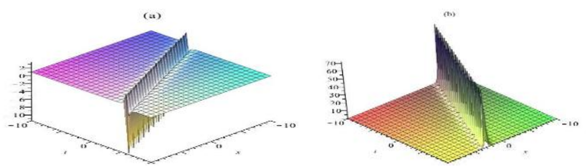

Figure 3: (a) 3-D plot of the plane-wave solution of u_3 with $-10 \leq x \leq 10$, $-10 \leq t \leq 10$, (b) 3-D plot of the singular cuspon of v_3 with $-10 \leq x \leq 10$, $-10 \leq t \leq 10$.

The exact solutions v_3 and v_5 are singular cuspon of the coupled Whitham-Broer-Kaup equation in the x-y plane. Shape of Eq. (25) (v_3) with $b_0 = 0$, $b_1 = \frac{1}{2}$, $b_2 = \frac{1}{2}$, $\tau = 0$, $\kappa = 1$, $c = \frac{1}{2}$, D = 2, y = 0, z = 0, y = 0, z = 0, and Eq. (29) (v_5) with $b_0 = \frac{1}{2}$, $b_1 = 0$, $b_2 = \frac{1}{2}$, $\tau = 0$, $\kappa = 0$, c = 1, D = 3, y = 0, z = 0 into the interval $-10 \leq x \leq 10$, $-10 \leq t \leq 10$ (see Figs. (3) and (5)).

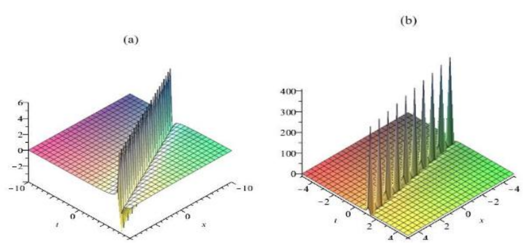

Figure 4: 3-D plot of the plane-waves solutions of u_4 and v_4 with $-10 \leq x \leq 10$, $-10 \leq t \leq 10$ respectively.

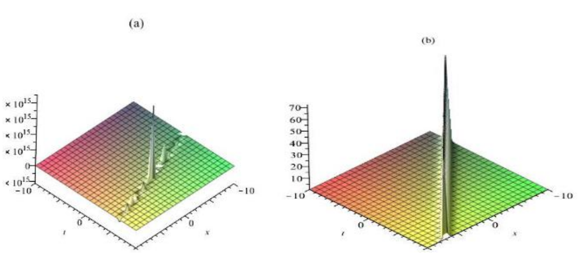

Figure 5: (a) 3-D plot of the plane-wave solution of u_5 with $-10 \leq x \leq 10$, $-10 \leq t \leq 10$, (b) 3-D plot of the singular cuspon of v_5 with $-10 \leq x \leq 10$, $-10 \leq t \leq 10$.

B. The new coupled KdV equation: The physical interpretation of the traveling wave solutions of the new coupled KdV will now be discussed. Different types of traveling waves have been deduced for the new coupled KdV equation, such as bell-type solitary waves, kink-type, periodic traveling waves, singular kink-type, singular cuspon type as well as plane-wave solutions.

Fig. 6 depicts the wave profiles corresponding to the solutions u_6, v_6, and w_6 for $a_0 = -\frac{1}{3}$, $a_1 = 1$, $a_2 = \frac{1}{3}$, $b_0 = 1$, $b_1 = \frac{2}{3}$, $b_2 = 0$, $d_0 = 1$, $d_1 = 1$, $d_2 = 1$, $\tau = 1$, $\kappa = 3$, c = 1, $D = \frac{1}{2}$, y = 0, z = 0 and $-5 \leq x \leq 5$, $-5 \leq t \leq 5$.

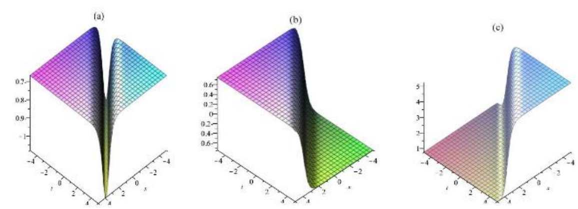

Figure 6: (a) 3-D plot of the bell-type solitary wave of u_6 with $-10 \leq x \leq 10$, $-10 \leq t \leq 10$, (b) and (c) 3-D plot of the kink-type soliton of v_6 and w_6 with $-10 \leq x \leq 10$, $-10 \leq t \leq 10$ respectively

From Fig. 6, we observe that the traveling wave solution u_6 is a bell-type solitary wave with infinite support or infinite tails. This soliton is characterized as a non-topological soliton. After the interaction, it was found that the amplitude and width of these waves decrease with increasing time t. We also note that in the x-y plane, the traveling wave solutions v_6 and w_6 represent kink-type soliton.

Fig. 7 presents the wave profiles corresponding to the solutions u_7, v_7 and w_7 which are the exact periodic traveling wave solutions of the new coupled KdV equation in the x-y plane. Shape of Eq. (45) (u_7) with $a_0 = -.4333333334$, $a_1 = \frac{1}{3}$, $a_2 = \frac{1}{3}$, $\tau = 0.7$, $\kappa = 1$, c = 1, D = 1, y = 0, z = 0. Eq. (46) (v_7) with $b_0 = 1$, $b_1 = 2$, $b_2 = 0$, $\tau = \frac{1}{2}$, $\kappa = 1$, c =

1, $D = 1$, $y = 0$, $z = 0$ and Eq. (47) (w_7) with $d_0 = 1$, $d_1 = 1$, $d_2 = 1$, $\tau = 1$, $\kappa = 1$, $c = 1$, $D = 2$, $y = 0$, $z = 0$ into the interval $-5 \leq x \leq 5$, $-5 \leq t \leq 5$.

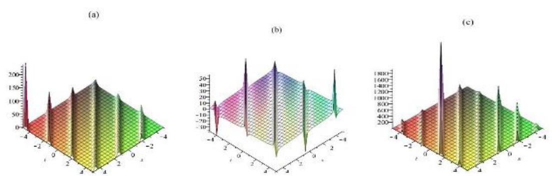

Figure 7: 3-D plot of the periodic traveling wave solutions of u_7, v_7 and w_7 with $-10 \leq x \leq 10$, $-10 \leq t \leq 10$ respectively.

Fig. 8 shows the wave profiles corresponding to the solutions u_8, v_8 and w_8 for $a_0 = -\frac{2}{3}$, $a_1 = \frac{1}{3}$, $a_2 = \frac{1}{3}$, $b_0 = 1$, $b_1 = 2$, $b_2 = 0$, $d_0 = 1$, $d_1 = 1$, $d_2 = 1$, $\tau = 0$, $\kappa = 1$, $c = 1$, $D = 2$, $y = 0$, $z = 0$ and $-5 \leq x \leq 5$, $-5 \leq t \leq 5$. From Fig. 8, we notice that the traveling wave solutions u_8 and w_8 represent singular kink-type solutions while the traveling wave solution v_8 represents the plane-wave solution.

Fig. 9 exhibits the wave profiles corresponding to the solutions u_9, v_9 and w_9 for $a_0 = -\frac{1}{3}$, $a_1 = \frac{2}{3}$, $a_2 = \frac{1}{3}$, $b_0 = 1$, $b_1 = 1$, $b_2 = 0$, $d_0 = 1$, $d_1 = 1$, $d_2 = 1$, $\tau = 1$, $\kappa = 2$, $c = 1$, $D = \frac{1}{2}$, $y = 0$, $z = 0$ and $-5 \leq x \leq 5$, $-5 \leq t \leq 5$. The exact solutions u_9 and w_9 represent singular cuspon while the exact solution v_9 represents the plane-wave solution. Cuspons are distinguished from other solitons by the presence of cusps at the crests of their solitons.

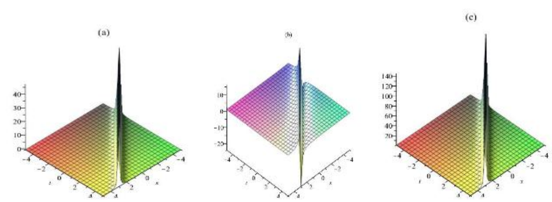

Figure 8: (a) and (c) 3-D plot of the singular Kink-type solutions of u_8 and w_8 with $-10 \leq x \leq 10$, $-10 \leq t \leq 10$ respectively, (b) 3-D plot of the plane-wave solution of v_8 with $-10 \leq x \leq 10$, $-10 \leq t \leq 10$.

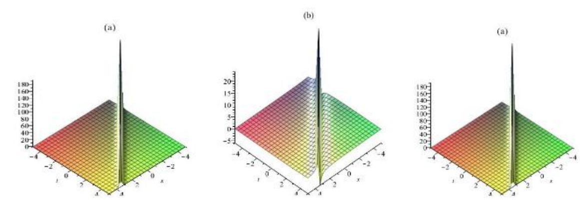

Figure 9: (a) and (c) 3-D plot of the singular cuspon of u_9 and w_9 with $-10 \leq x \leq 10$, $-10 \leq t \leq 10$ respectively, (b) 3-D plot of the plane-wave solution of v_9 with $-10 \leq x \leq 10$, $-10 \leq t \leq 10$.

5. Comparison

Several authors have employed various techniques to find the traveling wave solutions for the coupled Whitham-Broer-Kaup equation and the new coupled KdV equation. For example, Abbasbandy (Abbasbandy, 2007) employed the homotopy analysis method to obtain the traveling wave solutions of the coupled KdV equation; additionally, Rashidi et al. (Rashidi et al., 2008) utilized the homotopy analysis method to derive the solutions of the coupled Whitham-Broer-Kaup equations. We observed that the solutions derived by Abbasbandy (Abbasbandy, 2007) and Rashidi et al. (Rashidi et al., 2008) are different from the solutions obtained by applying the $\exp(-\Phi(\zeta))$ expansion method. The main advantage of the proposed approach is that these solutions are new and have not been published in any other literature. It is worth noting that these new solutions are obtained through very simple and easy calculations. Likewise, it can be demonstrated that the suggested method is much simpler than alternative methods for any nonlinear evolution equation.

6. Conclusion

In this study, the $\exp(-\Phi(\zeta))$ expansion method is successfully applied to derive exact solutions to the nonlinear coupled

Whitham-Broer-Kaup equation and the new coupled KdV equation. The traveling wave solutions were obtained in the form of exponential functions, hyperbolic functions, trigonometric functions, and rational functions. These solutions have many applications in quantum field theory, fluid mechanics, nonlinear optics, and plasma physics. By comparing the results of the $\exp(-\Phi(\zeta))$ expansion method with the results of the homotopy analysis method, we conclude that the solutions obtained are novel and have not been found elsewhere. We observe that the $\exp(-\Phi(\zeta))$ expansion technique is an easy, straightforward, and constructive method to obtain new traveling wave solutions. Results demonstrate that the $\exp(-\Phi(\zeta))$ expansion technique is very effective for solving many other NLEEs.

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